TIME ENCODING AND PERFECT RECOVERY OF BANDLIMITED SIGNALS

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ABSTRACT

A Time Encoding Machine is a real-time asynchronous mechanism for encoding amplitude information into a time sequence. We investigate the operating characteristics of a machine consisting of a feedback loop containing an adder, a linear filter and a Schmmitt trigger. We show how to recover the amplitude information of a bandlimited signal from the time sequence loss-free.

1. INTRODUCTION

A fundamental question arising in information processing is how to represent a signal as a discrete sequence. The classical sampling theorem ([6], [10]) calls for representing a bandlimited signal based on its samples taken at or above the Nyquist rate.

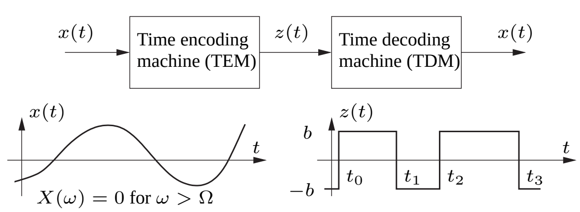

Fig. 1. Time Encoding and Decoding.

A time encoding of a bandlimited function $x(t), t \in \mathbb{R}$, is a representation of $x(t)$ as a sequence of increasing times $(t_k), k \in \mathbb{Z}$ (see Figure 1). Equivalently, the output of the encoder is a digital signal $z(t)$ that switches between two values $\pm b$ at times $t_k, k \in \mathbb{Z}$. Time encoding is an alternative to classical sampling and applications abound. In the field of neuroscience the representation of sensory information as a sequence of action potentials can be modeled as temporal encoding. The existence of a such a code was already postulated in [1]. Time encoding is also of great interest for the design and implementation of future analog to digital converters. Due to the ever decreasing size of integrated circuits and the attendant low voltage, high precision quantizers are more and more difficult to implement. These circuits provide increasing timing resolution, however, that a temporal code can take advantage of [9].

There are two natural requirements that a time encoding mechanism has to satisfy. The first is that the encoding should be implemented as a *real-time asynchronous* circuit. Secondly, the encoding mechanism should be *invertible*, that is the amplitude information can be recovered from the time sequence with arbitrary accuracy.

The encoding mechanism investigated in this paper satisfies both of these conditions. We show that a Time Encoding Machine (TEM) consisting of a feedback loop that contains an adder, a linear filter and a noninverting Schmitt trigger has the required properties. We also show how to build a non-linear inverse Time Decoding Machine (TDM) (see Figure 1) that perfectly recovers the amplitude information from the time sequence.

2. TIME ENCODING

The TEM investigated in this paper is depicted in Figure 2. The filter is assumed here to be an integrator. Clearly the amplitude information at the input of the TEM is represented as a time sequence at its output.

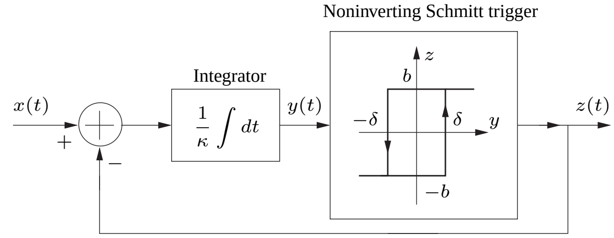

Fig. 2. An Example of a Time Encoding Machine

The basic principle of operation of the Time Encoding Machine is very simple. The bounded input signal $x(t)$, $|x(t)| \leq c < b$, is biased by a constant amount $+(-)b$ before being applied to the integrator. This bias guarantees that the integrator's output $y(t)$ is a positive (negative) increasing (decreasing) function of time. In steady state, there

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are two possible operating modes. In the first mode, the output of the TEM is in state $z(t) = -b$ and the input to the Schmitt trigger grows from $-\delta$ to δ. When the output of the integrator reaches the maximum value δ, a transition of the output $z(t)$ from $-b$ to $+b$ is triggered and the feedback becomes negative. In the second mode of operation, the TEM is in state $z(t) = b$ and the integrator output steadily decreases from δ to $-\delta$. When the maximum negative value $-\delta$ is reached $z(t)$ will reverse to $-b$. Thus, while the transition times of the output $z(t)$ are non-uniformly spaced, the amplitude of the output signal remains constant. Therefore, a transition of the output from $-b$ to b or vice-versa takes place every time the integrator output reaches the triggering mark δ or $-\delta$ (called quanta). The time when this quanta is achieved depends on the signal as well as on the design parameter b. Hence, the Time Encoding Machine is mapping amplitude information into timing information. It achieves this by a signal-dependent sampling mechanism.

2.1. Stability and the Compensation Principle

In Figure 2, $\kappa,\ \delta,\ b$ are strictly positive real numbers and $x = x(t)$ is a Lebesgues measurable function that models the input signal to the TEM for all $t,\ t \in \mathbb{R}$. The output of the integrator is given by:

$$y(t) = y(t_0) + \frac{1}{\kappa}\int_{t_0}^{t}[x(u) - z(u)]du, \qquad (1)$$

for all $t,\ t \geq t_0$. Note that $y = y(t)$ is a continous increasing (decreasing) function whenever the value of the feedback is positive (negative). Here, $z : \mathbb{R} \rightarrow \{-b, b\}$ for all $t, t \in \mathbb{R}$, is the function corresponding to the output of the TEM in Figure 2. z switches between two values $+b$ and $-b$ at a set of trigger times (t_k), for all $k,\ k \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers and $z(t_0) = -b$ by convention.

Remark 1 Informally, the information of the input $x(t)$ is carried by the signal amplitude whereas the information of the output signal $z(t)$ is carried by the trigger times. A fundamental question, therefore, is whether the Time Encoding Machine encodes information loss-free. Loss-free encoding means that $x(t)$ can be perfectly recovered from $z(t)$.

Lemma 1 (Stability) *For all input signals $x = x(t),\ t \in \mathbb{R}$, with $|x(t)| \leq c < b$ the TEM is stable, i.e., $|y(t)| \leq \delta$, for all $t,\ t \in \mathbb{R}$. The output z is given by $z(t) = b\ (-1)^{k+1}$ for all $t_k \leq t < t_{k+1}$, $t \in \mathbb{R}$, where the set of trigger times (t_k), $k \in \mathbb{Z}$, is obtained from the recursive equation*

$$\int_{t_k}^{t_{k+1}} x(u)du = (-1)^k[-b(t_{k+1} - t_k) + 2\kappa\delta]. \qquad (2)$$

for all $k,\ k \in \mathbb{Z}$.

Proof: Due to the operating characteristic of the Schmitt trigger, y reaches the value δ if the feedback is b or the value $-\delta$ if the feedback is $-b$ for any arbitrary initial value of the integrator. Therefore, without loss of generality we can assume that for some initial condition $t = t_0$ we have $(y, z) = (-\delta, -b)$ and the Time Encoding Machine is described in a small neighborhood of t_0 by:

$$-\delta + \frac{1}{\kappa}\int_{t_0}^{t}[x(u) + b]du = \delta. \qquad (3)$$

Since the left hand side is a continuously increasig function, there exists a time $t = t_1$ such that the equation above holds. Similarly starting with $(y, z) = (\delta, b)$ at time t_1 the equation:

$$\delta + \frac{1}{\kappa}\int_{t_1}^{t}[x(u) - b]du = -\delta. \qquad (4)$$

is satisfied for some $t = t_2$. Thus, the sequence (t_k), $k \in \mathbb{Z}$, defined by the equations (2) uniquely describe the (output) function $z = z(t)$, for all $t,\ t \in \mathbb{R}$, and $|y| \leq \delta$ by construction.

Corollary 1 (Upper and Lower Bounds for Trigger Times) *For all input signals $x = x(t),\ t \in \mathbb{R}$, with $|x(t)| \leq c < b$, the distance between consecutive trigger times t_k and t_{k+1} is given by:*

$$\frac{2\kappa\delta}{b + c} \leq t_{k+1} - t_k \leq \frac{2\kappa\delta}{b - c}, \qquad (5)$$

for all $k,\ k \in \mathbb{Z}$.

Proof: By applying the mean value theorem to the term on the left hand side of equation (2) we have

$$(-1)^k x(\xi_k)(t_{k+1} - t_k) = -b(t_{k+1} - t_k) + 2\kappa\delta, \qquad (6)$$

where $\xi_k \in [t_k, t_{k+1}]$. Solving for $t_{k+1} - t_k$ and noting that $|x(t)| \leq c$ we obtain the desired result. The bound is achieved for a constant input $x(t) = c$.

Lemma 2 (The Compensation Principle)

$$\int_{t_l}^{t_{l+2}} x(u)du = \int_{t_l}^{t_{l+2}} z(u)du, \qquad (7)$$

for all $l \in \mathbb{Z}$.

Proof: The desired result is obtained by adding equations (2) for $k = l$ and $k = l + 1$.

Remark 2 If $x(t)$ is a continuous function, there exists a $\xi_k \in [t_k, t_{k+2}]$, $k \in \mathbb{Z}$, such that:

$$x(\xi_k)(t_{k+2} - t_k) = (-1)^k[-b(t_{k+1} - t_k) + b(t_{k+2} - t_{k+1})], \qquad (8)$$

$i.e.$, the sample $x(\xi_k)$ can be explicitly recovered from information contained in the process $z(t)$, $t_k \leq t \leq t_{k+2}$, $k \in \mathbb{Z}$. Intuitively, therefore, any class of input signals that can be recovered from its samples can also be recovered from $z(t)$. Note also that the Compensation Principle provides for an estimate of the amplitude of the input signal $x(t)$ on a very small time scale that does not explicitly depend on $\kappa\delta$.

Remark 3 The Compensation Principle can be easily extended to subsets of or to the entire real line. Thus the DC component of the input can be recovered from $z(t)$ even for non-bandlimited input signals $x(t), t \in \mathbb{R}$.

3. PERFECT RECOVERY

A Time Decoding Machine has the task of recovering the signal $x = x(t)$, $t \in \mathbb{R}$, from $z = z(t)$, $t \in \mathbb{R}$, or a noisy version of the same. Here we will focus on the recovery of the original signal x based on z only. We shall show that a perfect recovery is possible, that is, the input signal x can be recovered from z without any loss of information.

Informally, the length of the interval between two consecutive trigger times of $z(t)$ provides an estimate of the integral of $x(t)$ on the same interval. This estimate can be used in conjunction with the bandlimited assumption on x to obtain a perfect reconstruction of the signal even though the trigger times are irregular. As expected, the interval between two consecutive trigger times has to be smaller then the distance between the uniformly spaced samples in the classical sampling theorem [6], [10].

The mathematical methodology used here is based on the theory of frames [3]. We shall construct a linear operator on L^2, the space of square integrable functions defined on $\mathbb{R}$, and by starting from a good initial guess followed by successive interations, obtain sucessive approximations that converge in the appropriate norm to the original signal x.

Let us assume that $x = x(t), t \in \mathbb{R}$, is a signal bandlimited to $[-\Omega, \Omega]$ and let the operator $\mathcal{A}$ be given by:

$$\mathcal{A}x = \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} x(u)du \, g(t - s_k), \tag{9}$$

where $g(t) = sin(\Omega t)/\pi t$ and $s_k = (t_{k+1} + t_k)/2$.

The realization of the operator $\mathcal{A}$ above is highly intuitive. Dirac-delta pulses are generated at times $(t_{k+1}+t_k)/2$ with weight $\int_{t_k}^{t_{k+1}} x(u)du$ and then passed through an ideal low pass filter with unity gain for $\omega \in [-\Omega, \Omega]$ and zero otherwise. The values of $\int_{t_k}^{t_{k+1}} x(u)du$ are available at the TDM through equation (2).

Let $x_l = x_l(t)$, $t \in \mathbb{R}$, be a sequence of bandlimited functions defined by the recursion:

$$x_{l+1} = x_l + \mathcal{A}(x - x_l), \tag{10}$$

for all l, $l \in \mathbb{Z}$, with the initial condition $x_0 = \mathcal{A}x$.

Note that since the distance between two consecutive trigger times is bounded by $2\kappa\delta/(b - c)$ (see equation (5)),

$$\| I - \mathcal{A} \| \leq r, \tag{11}$$

where $r = \frac{2\kappa\delta}{b-c} \frac{\Omega}{\pi}$ [4].

Theorem 1 (Operator Formulation) *Let $x = x(t), t \in \mathbb{R}$, be a bounded signal $|x(t)| \leq c < b$ bandlimited to $[-\Omega, \Omega]$. Let $z = z(t)$, $t \in \mathbb{R}$, be the ouput of a Time Encoding Machine with integrator constant κ and Schmitt trigger parameters (δ, b). If $\kappa\delta \leq \frac{b-c}{2} \frac{\pi}{\Omega}$, the signal x can be perfectly recovered from z as*

$$x(t) = \lim_{l\to\infty} x_l(t), \tag{12}$$

and

$$\| x - x_l \| \leq r^{l+1} \| x \| . \tag{13}$$

Proof: By induction we can show that

$$x_l = \sum_{k=0}^{l} (I - \mathcal{A})^k \mathcal{A}x. \tag{14}$$

Since $\| I - \mathcal{A} \| \leq r < 1$,

$$\lim_{l\to\infty} x_l = \sum_{k\in\mathbb{N}} (I - \mathcal{A})^k \mathcal{A}x = \mathcal{A}^{-1}\mathcal{A}x = x. \tag{15}$$

Also,

$$\begin{aligned} x - x_l &= \sum_{k\geq l+1} (I - \mathcal{A})^k \mathcal{A}x = (I - \mathcal{A})^{l+1} \sum_{k\in\mathbb{N}} (I - \mathcal{A})^k \mathcal{A}x \\ &= (I - \mathcal{A})^{l+1} \mathcal{A}^{-1}\mathcal{A}x = (I - \mathcal{A})^{l+1} x, \end{aligned} \tag{16}$$

and, therefore, $\| x - x_l \| \leq r^{l+1} \| x \|$.

Let us define $\mathbf{g} = [g(t - s_k)]^T$, $\mathbf{q} = [\int_{t_k}^{t_{k+1}} x(u) \, du]$ and $\mathbf{G} = [\int_{t_l}^{t_{l+1}} g(u - s_k) \, du]$. We have the following

Theorem 2 (Matrix Formulation) *Under the assumptions of Theorem 1 the bandlimited signal x can be perfectly recovered from z as*

$$x(t) = \lim_{l\to\infty} x_l(t) = \mathbf{g}\mathbf{G}^{-1}\mathbf{q}. \tag{17}$$

where $\mathbf{G}^{-1}$ denotes the pseudo-inverse of $\mathbf{G}$. Furthermore,

$$x_l(t) = \mathbf{g}\mathbf{P}_l\mathbf{q}, \tag{18}$$

where $\mathbf{P}_l$ is given by

$$\mathbf{P}_l = \sum_{k=0}^{l} (\mathbf{I} - \mathbf{G})^k. \tag{19}$$

Proof: By induction. Since $x_0(t) = \mathbf{gq}$ with $\mathbf{P}_0 = \mathbf{I}$, we assume that $x_l(t) = \mathbf{gP}_l\mathbf{q}$ with $\mathbf{P}_l = \sum_{k=0}^{l}(\mathbf{I} - \mathbf{G})^k$. We have

$$x_{l+1}(t) = \mathbf{g}(\mathbf{P}_l + \mathbf{I} - \mathbf{GP}_l)\mathbf{q} = \mathbf{gP}_{l+1}\mathbf{q}. \qquad (20)$$

The convergence of the sum for $\mathbf{P}_l$ is guaranteed by Theorem 1 and $\mathbf{GP}_\infty = \mathbf{I}$.

Let $h(t) = \sin(\Omega t)/\Omega t$, $t \in \mathbb{R}$, and $\mathbf{H} = [\int_{t_l}^{t_{l+1}} h(t - kT)]$ with $T = \pi/\Omega$.

Theorem 3 (Change of Frame) *Under the assumptions of Theorem 1, if* $t_l < kT < t_{l+1}$ *for all* $k \in \mathbb{Z}$ *and some* $l \in \mathbb{Z}$

$$\mathbf{p} = \mathbf{H}^{-1}\mathbf{q}, \qquad (21)$$

where $\mathbf{p} = [x(kT)]$, $k \in \mathbb{Z}$, *and* $\mathbf{H}^{-1}$ *is the inverse of* $\mathbf{H}$.

Proof: Integrating both sides of the classical sampling representation of bandlimited signals

$$x(t) = \sum_{k\in\mathbb{Z}} x(kT)\, h(t - kT), \qquad (22)$$

we obtain the desired result from

$$\int_{t_l}^{t_{l+1}} x(t)\, dt = \sum_{k\in\mathbb{Z}} x(kT) \int_{t_l}^{t_{l+1}} h(t - kT). \qquad (23)$$

4. EXAMPLE

The mathematical formulation of the previous section assumes that the dimensionality of the matrices and vectors used is infinite. In simulations, however, only a finite time window can be used. We briefly investigate three different implementations of the TDM in the finite dimensional case that are based on the (i) recursive equation (18), (ii) closed form formula (17), and (iii) change of frame formula (21).

In all our simulations, the input signal is given by (22) where the samples $x(T)$ through $x(12T)$, respectively, are given by -0.394103, 0.375745, 0.416555, 0.198506, -0.55382, 0.0405288, 0.583311, 0.278091, -0.135832, -0.292735, -0.223741, -0.585826, $x(kT) = 0$, for $k \le 0$ and $k > 12$ and $T = \pi/\Omega = 1.25$ ms. Fig. 3(a) shows $x(t)$ together with the time window used for simulations. Fig. 3(b) shows the simulation results for $y(t)$ and $z(t)$ with $\delta = 0.55$, $b = 1$, and $\kappa = 318.31$ μs. The 40 trigger times of $z(t)$ shown were determined with high accuracy using (2).

(i) The error signals shown by Fig. 4(a) are defined as $e_l = e_l(t) = x_l(t) - x(t)$, where $x_l(t)$ was calculated based on (18). Instead of applying (19) directly we used the recursion $\mathbf{P}_{l+1} = \mathbf{I} + \mathbf{P}_l(\mathbf{I} - \mathbf{G})$ and calculated $x_l(t)$ iteratively. As shown, $e_l(t)$ decreases in agreement with Theorem 1, since with the parameters introduced $r = 0.7115 < 1$. We

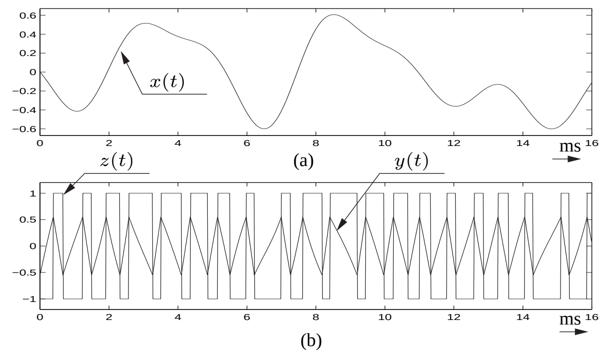

Fig. 3. Overall bandlimited input signal $x(t)$ (a), integrator output signal $y(t)$ and the TEM output signal $z(t)$ (b).

note (not shown in the figure) that $\max_t(e_{200}) = 1.52 \times 10^{-4}$.

(ii) Although the matrix $\mathbf{G}$ in (17) is singular, perfect recovery can be achieved since $\mathbf{G}^{-1}$, the pseudo-inverse of $\mathbf{G}$, exists. The corresponding error signal defined as $\mathbf{gG}^{-1}\mathbf{q} - x(t)$ is shown by the solid line of Fig. 4(b). The small error is due to the finite precision used.

(iii) The dashed line of Fig.4(b) shows the error when the bandlimited signal $x(t)$ is recovered using the sampling representation (22) and the samples $x(kT)$ are obtained from (21). The t_l's selected represent the set of closest pair of trigger times around $kT - T/2$, $k \in \mathbb{Z}$. All other trigger times are dropped. Again, the small error is due to numerical inaccuracies. We note that matrix $\mathbf{H}$ turned out to be not only invertible but well-conditioned as well.

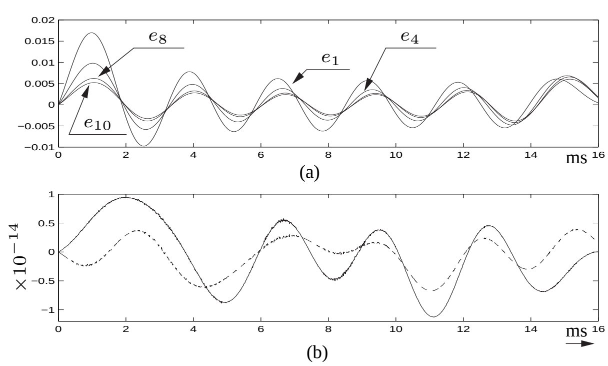

Fig. 4. Approximating signals using iteration (a), overall error signals using closed formulas (b).

5. RELATIONSHIP TO OTHER MODULATION SCHEMES

The Time Decoding Machine and the demodulator for Frequency Modulation (FM) [2] operate on a signal that has the same information structure.

Recall that FM demodulation is achieved by finding the times t such that:

$$\sin(\omega t + \eta \int_{t_0}^{t} x(u)du + \phi) = 0 \quad (24)$$

where ω is the modulation frequency and η is the modulation index. Therefore,

$$\int_{t_k}^{t_{k+1}} x(u)du = -\frac{\omega}{\eta}(t_{k+1} - t_k) + \frac{\pi}{\eta}. \quad (25)$$

The above equation and equation (2) have the same basic structure. Hence an FM modulated signal x can be perfectly recovered from the sequence of times (t_k), $k \in \mathbb{Z}$ using the Time Decoding Machine. These observations establish a bridge to non-uniform sampling methods previously applied to improve the performance of FM and other nonlinear modulators [8].

The Time Encoding Machine also models an Asynchronous Sigma-Delta modulator [5] and, therefore, the latter is invertible. Past attempts at building Sigma-Delta demodulators have led to low accuracy in signal recovery [9]. This is because of the linear structure of these demodulators.

6. CONCLUSIONS

We showed that a simple Time Encoding Machine can be used for generating time codes for arbitrary bandlimited signals. The TEM consists of a feedback loop that contains an adder, a filter and a noninverting Schmitt trigger. We derived a simple condition that guarantees that the amplitude of the bandlimited signal can be recovered from the time sequence loss-free. We also presented algorithms for perfect recovery and briefly investigated their performance.

TEMs can easily be incorporated into current digital systems by measuring the trigger-times $(t_k), k \in \mathbb{Z}$. The analysis of such a system will be presented elsewhere [7].

Finally, it has not escaped the authors that the Time Encoding Machine can be used as a neuro-modulator with perfect information recovery. Therefore, such a modulator can be applied to image and auditory neural coding.

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