SENSITIVITY ANALYSIS OF TIME ENCODED BANDLIMITED SIGNALS

Aurel A. Lazar

Department of Electrical Engineering
Columbia University, New York, NY 10027
aurel@ee.columbia.edu

László T. Tóth

Dept. of Telecom. and Media Informatics
Budapest Univ. of Technology and Economics
Budapest, Hungary H-1117, tothl@tmit.bme.hu

ABSTRACT

A Time Encoding Machine consisting of a feedback loop containing an adder, an integrator and a Schmitt trigger encodes amplitude information into a time sequence. We demonstrate how to construct a Time Decoding Machine that perfectly recovers the amplitude information from the time sequence and is trigger parameter *insensitive*.

We derive bounds on the error in signal recovery introduced by the quantization of the time sequence. We compare these with the recovery error introduced by the quantization of the amplitude of the bandlimited signal when irregular sampling is employed. Under Nyquist-type rate conditions, quantization of a bandlimited signal in the time and amplitude domains are shown to be largely equivalent methods of information representation.

1. INTRODUCTION

A time encoding of a bandlimited function $x(t), t \in \mathbb{R}$, is a representation of $x(t)$ as a sequence of strictly increasing times $(t_k), k \in \mathbb{Z}$, where $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and integers, respectively. Alternately, the bandlimited function is encoded as a digital signal $z(t)$ that switches between two values ± 1 at times $t_k, k \in \mathbb{Z}$. A Time Encoding Machine (TEM) is a real-time asynchronous mechanism for encoding amplitude information into a time sequence. A Time Decoding Machine (TDM) recovers the amplitude information from the time sequence.

In [3] a TEM amenable to nano-scale integration was investigated. The machine consists of a feedback loop that contains an adder, a linear filter and a non-inverting Schmitt trigger (see Figure 1). It was shown there, that the amplitude information of the encoded signal $x(t)$, $t \in \mathbb{R}$, can be perfectly recovered from the sequence (t_k), $k \in \mathbb{Z}$, provided that the difference between any two consecutive values of the time sequence is bounded by the inverse of the Nyquist rate. This has established time encoding as an information representation modality for bandlimited signals.

In practice, the question of *sensitivity* of the recovery algorithm with respect to parameter variation of the TEM is of outmost importance. In this paper we investigate the sensitivity of signal recovery with respect to the Schmitt trigger parameter δ as well as with respect to the number N of bits used to quantize the values of the trigger times.

Through simple simulations we demonstrate that the TDM that implements the perfect recovery algorithm is highly sensitive to a broad range of values of δ. Based on the simple compensation principle of [3] we provide a perfect recovery algorithm that is δ-insensitive.

We evaluate the error introduced by the quantization of the time sequence and derive bounds on the recovery error. We compare these with the recovery error introduced by the quantization of the amplitude of an arbitrary bandlimited signal when irregular sampling is employed. Under Nyquist-type rate conditions, quantization of a bandlimited signal in the time and amplitude domains are shown to be largely equivalent methods of information representation.

This paper is organized as follows. Time encoding and perfect recovery algorithms are reviewed in section 2. Section 3 investigates the sensitivity of the recovery algorithm with respect to the Schmitt trigger parameter δ. The Compensation Principle is used to build a δ-insensitive recovery algorithm. The effect of quantization of the trigger times on signal recovery is discussed in section 4. Finally, in section 5 the effects of quantization in the time and amplitude domains on the recovery of bandlimited signals are compared.

2. TIME ENCODING AND PERFECT RECOVERY

The TEM considered in this paper is an (essentially) equivalent version of the one investigated in [3] (see Figure 1). The input signal to the TEM is modelled as a Lebesgues measurable function $x = x(t)$, $t, t \in \mathbb{R}$, in L^2. Furthermore, x is bounded, $|x(t)| \leq c < 1$, and bandlimited to $[-\Omega, \Omega]$.

The output of the TEM is a function z taking two values $z : \mathbb{R} \to \{-1, 1\}$ for all $t, t \in \mathbb{R}$, with transition times (t_k),

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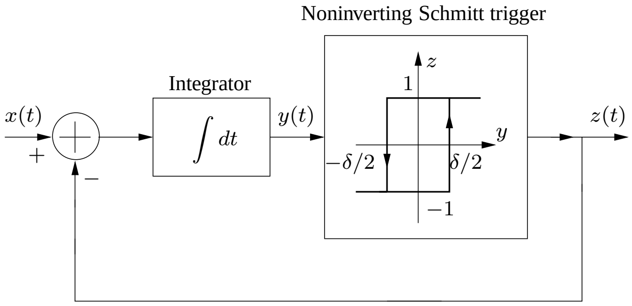

Fig. 1. The Time Encoding Machine

$k \in \mathbb{Z}$, generated by the recursive equations

$$\int_{t_k}^{t_{k+1}} x(u)du = (-1)^k[\delta - (t_{k+1} - t_k)], \tag{1}$$

for all $k,\ k \in \mathbb{Z}$. Intuitively, these equations map the amplitude information of the signal $x(t)$, $t \in \mathbb{R}$, into the time sequence (t_k), $k \in \mathbb{Z}$, and implicitly define a *signal-dependent* sampling mechanism.

Let $x_l = x_l(t)$, $t \in \mathbb{R}$, be a sequence of bandlimited functions defined by the recursion:

$$x_{l+1} = x_l + \mathcal{A}(x - x_l), \tag{2}$$

for all $l,\ l \in \mathbb{N}$, with the initial condition $x_0 = \mathcal{A}x$, where the operator $\mathcal{A}$ is given by:

$$\begin{aligned} \mathcal{A}x &= \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} x(u)du\; g(t - s_k) \\ &= \sum_{k\in\mathbb{Z}} (-1)^k[\delta - (t_{k+1} - t_k)]\; g(t - s_k), \end{aligned} \tag{3}$$

with $g(t) = sin(\Omega t)/\pi t$ and $s_k = (t_{k+1} + t_k)/2$. In what follows I and $\mathbf{I}$ will denote the identity operator and the identity matrix, respectively. In [3] the following TDM perfect recovery algorithm was derived (The most general recovery result only requires that the average number of trigger times is bounded by the inverse of the Nyquist rate [2]. However, this result lacks operational significance in our setting.):

Theorem 1 (Operator Formulation) *If $r = \frac{\delta}{1-c}\frac{\Omega}{\pi} < 1$, the bandlimited signal x can be perfectly recovered from the trigger times (t_k), $t \in \mathbb{Z}$, as*

$$\lim_{l\to\infty} x_l(t) = x(t) = \sum_{k\in\mathbb{N}} (I - \mathcal{A})^k \mathcal{A}x, \tag{4}$$

and

$$\| x - x_l \| \leq r^{l+1} \| x \| . \tag{5}$$

With $\mathbf{g} = [g(t - s_k)]$ and $\mathbf{q} = [(-1)^k(\delta + t_k - t_{k+1})]$ denoting vectors and $\mathbf{G} = [\int_{t_l}^{t_{l+1}} g(u - s_k)\, du]$ denoting a matrix, we have the following

Theorem 2 (Matrix Formulation) *If $r = \frac{\delta}{1-c}\frac{\Omega}{\pi} < 1$, the bandlimited signal x can be perfectly recovered from the trigger times (t_k), $t \in \mathbb{Z}$, as*

$$x(t) = \lim_{l\to\infty} x_l(t) = \mathbf{g}^T\mathbf{G}^+\mathbf{q}. \tag{6}$$

where $\mathbf{G}^+$ denotes the pseudo-inverse of $\mathbf{G}$. Furthermore, $x_l(t) = \mathbf{g}^T\mathbf{P}_l\mathbf{q}$, where $\mathbf{P}_l = \sum_{k=0}^{l}(\mathbf{I} - \mathbf{G})^k$.

Remark 1 If $\mathbf{c} = [c_k]$ is the vector defined by $\mathbf{c} = \mathbf{G}^+\mathbf{q}$ then the recovery formula (6) becomes

$$x(t) = \sum_{k\in\mathbb{Z}} c_k g(t - s_k). \tag{7}$$

Therefore, the recovery algorithm given by equation (6) has a very simple interpretation. Dirac–delta pulses generated at times s_k with weight c_k are passed through a low pass filter with unity gain for $\omega \in [-\Omega, \Omega]$ and zero otherwise. For a precise definition and motivation of the pseudo-inverse the reader is referred to [5].

3. RECOVERY SENSITIVITY WITH RESPECT TO δ

In this section we will first demonstrate the high sensitivity of the perfect recovery algorithm with respect to implementation errors of the parameter δ in the TDM. We will then demonstrate how this can be overcome and advance a δ-insensitive recovery algorithm.

3.1. δ with a fixed error ε at the TDM

The model considered in this section is based on the premise that the TEM is employing δ and the TDM implements $\delta+\varepsilon$ and has exact knowledge of the trigger times. The reconstruction algorithm consistently generates an error signal e given by:

$$e(t) = x(t) - \hat{x}(t) = \sum_{k\in\mathbb{N}} (I - \mathcal{A})^k \varepsilon \sum_{l\in\mathbb{Z}} g(t - s_l), \tag{8}$$

where $\hat{x}$ is the output of a TDM that uses $\delta + \epsilon$ for recovery.

In what follows we define a mean-square error measure $\mathcal{E}^2$ as

$$\mathcal{E}^2 = \lim_{n\to\infty} \frac{1}{2nT_{min}} \| e1_{[-nT_{min}, nT_{min}]} \|^2, \tag{9}$$

where 1 denotes the indicator function,

$$\| e1_{[-nT_{min}, nT_{min}]} \|^2 = \int_{\mathbb{R}} e^2(u) 1_{[-nT_{min}, nT_{min}]}(u)du, \tag{10}$$

and $T_{min} = \min_{k\in\mathbb{Z}} T_k$ with $T_k = t_{k+1} - t_k$.

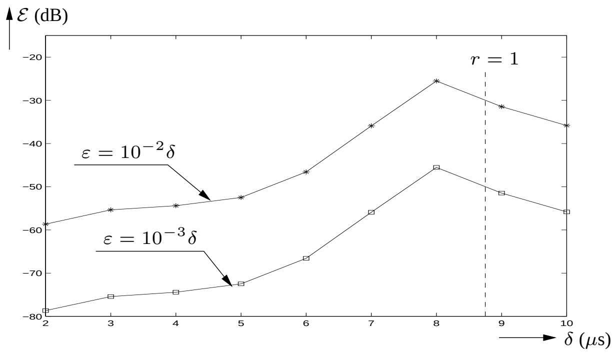

Fig. 2. The dependence of $\mathcal{E}$ on δ parameterized by $\varepsilon = 10^{-2}\delta$ (stars) and $\varepsilon = 10^{-3}\delta$ (squares).

Example 1 A sample of the dependance of the mean square recovery error on δ parameterized by ε is shown in Figure 2. In all our simulations, the input signal is given by $x(t) = \sum_{k\in\mathbb{Z}} x(kT)g(t-kT)$ where the samples $x(T)$ through $x(12T)$, are respectively, -0.1961, 0.186965, 0.207271, 0.0987736, -0.275572, 0.0201665, 0.290247, 0.138374, -0.067588, -0.145661, -0.11133, -0.291498, $x(kT) = 0$, for $k \leq 0$ and $k > 12$; $c = 0.3$, $\Omega = 2\pi \cdot 40$ kHz and $T = \pi/\Omega = 12.5$ μs. The evaluation of the trigger times was carried out in the interval $-2T \leq t \leq 15T$.

3.2. δ-Insensitive Recovery Algorithm

As shown in Figure 2, the implementation of the TDM recovery algorithm given in Theorem 2, is highly sensitive to the exact knowledge of the parameter δ. Remedy is provided by the following [3]

Lemma 1 (The Compensation Principle)

$$\int_{t_l}^{t_{l+2}} x(u)du = (-1)^l[(t_{l+2} - t_{l+1}) - (t_{l+1} - t_l)], \quad (11)$$

for all $l \in \mathbb{Z}$.

Proof: The desired result is obtained by adding equations (1) for $k = l$ and $k = l + 1$.

The Compensation Principle suggests the construction of an operator of the form

$$\begin{aligned} \mathcal{B}x &= \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+2}} x(u)du \; f_{k+1}(t) \\ &= \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} x(u)du \; [f_k(t) + f_{k+1}(t)]. \end{aligned} \quad (12)$$

The operators $\mathcal{A}$ and $\mathcal{B}$ are identical provided that $\mathbf{g} = \mathbf{B}^T\mathbf{f}$, where $\mathbf{f} = [f_k]$ and the elements of the matrix $\mathbf{B}$ are given by $[\mathbf{B}]_{k,l} = 1$ for $k = l$ or $k = l + 1$ and zero otherwise. Note that, the inverse of $\mathbf{B}$ is given by $[\mathbf{B}^{-1}]_{k,l} = (-1)^{k-l}$ for $k \geq l$ and zero otherwise. Note also that

$$\mathbf{Bq} = [(-1)^k(t_{k+2} - 2t_{k+1} + t_k)]$$

does not explicitly depend on δ.

Theorem 3 (δ-insensitive recovery algorithm - matrix form) *If* $r = \frac{\delta}{1-c} \cdot \frac{\Omega}{\pi} < 1$, *the bandlimited signal* x *can be perfectly recovered from its associated trigger times* (t_k), $k \in \mathbb{Z}$, *without explicit knowledge of the parameter* δ *as*

$$x(t) = \lim_{l\to\infty} x_l(t) = \mathbf{g}^T \cdot \mathbf{B}^{-1}(\mathbf{BGB}^{-1})^+ \cdot \mathbf{Bq}. \quad (13)$$

Furthermore,

$$x_l(t) = \mathbf{g}^T \cdot \mathbf{B}^{-1}\mathbf{Q}_l \cdot \mathbf{Bq}, \quad (14)$$

where $\mathbf{Q}_l$ *is given by*

$$\mathbf{Q}_l = \sum_{k=0}^{l} [\mathbf{I} - \mathbf{BGB}^{-1}]^k. \quad (15)$$

Proof: Using the notation of Theorem 2, x_l can be rewritten as

$$x_l(t) = \mathbf{g}^T\mathbf{P_l q} = \mathbf{g}^T \cdot \mathbf{B^{-1}}(\mathbf{BP_lB^{-1}}) \cdot \mathbf{Bq}.$$

Since

$$\mathbf{BP_lB^{-1}} = \mathbf{B}\sum_{k=0}^{l}(\mathbf{I} - \mathbf{G})^k\mathbf{B^{-1}} = \sum_{k=0}^{l}(\mathbf{I} - \mathbf{BGB^{-1}})^{\mathbf{k}}$$

we have (see [5] for the introduction of the pseudo-inverse)

$$\begin{aligned} x(t) &= \lim_{l\to\infty} \mathbf{g}^T \cdot \mathbf{B}^{-1} \sum_{k=0}^{l} (\mathbf{I} - \mathbf{BGB^{-1}})^{\mathbf{k}} \cdot \mathbf{Bq} \\ &= \mathbf{g}^T \cdot \mathbf{B}^{-1}(\mathbf{BGB}^{-1})^+ \cdot \mathbf{Bq}. \end{aligned} \quad (16)$$

Example 2 The δ-insensitive recovery algorithm achieves perfect recovery provided that $r < 1$. Simulation results for the δ-sensitive and δ-insensitive recovery algorithms are shown in Figure 3 and are denoted by stars and squares, respectively. The dotted vertical line corresponds to the value of δ for which $r = 1$.

4. RECOVERY SENSITIVITY WITH RESPECT TO TIME QUANTIZATION

In this section we shall assume that the sequence of trigger times (t_k), $k \in \mathbb{Z}$, is measured with finite precision and the actual values available for recovery are $\hat{t}_k$, $k \in \mathbb{Z}$. We shall denote by $T_k = t_{k+1} - t_k$ and $\hat{T}_k = \hat{t}_{k+1} - \hat{t}_k$ for all $k \in \mathbb{Z}$.

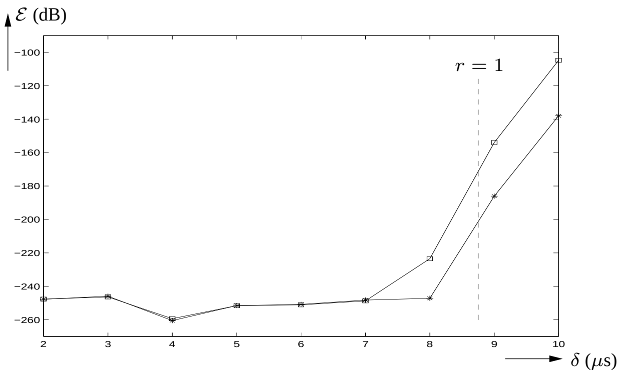

Fig. 3. Mean square error for the δ-sensitive (stars) and δ-insensitive algorithms (squares).

4.1. An Upper Bound on a Measure of Error Recovery

The key point of our analysis is the observation that, if the condition $\max_k(\hat{T}_k < T)$ is satisfied, then

$$x = \sum_{k\in\mathbb{N}}(I - \hat{\mathcal{A}})^k \hat{\mathcal{A}} x,$$

where $\hat{\mathcal{A}}$ is defined by

$$\hat{\mathcal{A}} x = \sum_{k\in\mathbb{Z}} \int_{\hat{t}_k}^{\hat{t}_{k+1}} x(u)du \; g(t - \hat{s}_k) \tag{17}$$

and $\hat{s}_k = (\hat{t}_k + \hat{t}_{k+1})/2$. Since the reconstructed signal is given by

$$\hat{x} = \sum_{k\in\mathbb{N}}(I - \hat{\mathcal{A}})^k \sum_{l\in\mathbb{Z}}[(-1)^l(-\hat{T}_l + \delta)]g(t - \hat{s}_l),$$

the error signal amounts to

$$e(t) = \sum_{k\in\mathbb{N}}(I - \hat{\mathcal{A}})^k \sum_{l\in\mathbb{Z}} \epsilon_l g(t - \hat{s}_l), \tag{18}$$

where

$$\epsilon_k = (-1)^k(-\hat{T}_k + \delta) - \int_{\hat{t}_k}^{\hat{t}_{k+1}} x(u)du. \tag{19}$$

Proposition 1 *Assuming that the quantization error $d_k = \hat{T}_k - T_k$, $k \in \mathbb{Z}$, can be modelled as a sequence of i.i.d. random variables on $[-\Delta/2, \Delta/2]$, the expected MSE is bounded by:*

$$\mathbb{E}\{\mathcal{E}^2\} \leq \frac{1+c}{\delta T} \cdot (\frac{1+c}{1-r})^2 \cdot \frac{\Delta^2}{12} \, . \tag{20}$$

Proof: See [4].

5. A COMPARISON OF TIME AND AMPLITUDE QUANTIZATION

In this section we highlight the relationship between time encoding and irregular sampling, i.e., between two information representations of a bandlimited signal as a discrete time and a discrete amplitude sequence, respectively.

5.1. Relationship to Irregular Sampling

In what follows we shall assume that the irregular samples $(x(s_k))$, $k \in \mathbb{Z}$, are available for signal reconstruction. $x_l = x_l(t)$, $t \in \mathbb{R}$, will denote a sequence of bandlimited functions defined by the recursion:

$$x_{l+1} = x_l + \mathcal{S}(x - x_l), \tag{21}$$

for all l, $l \in \mathbb{N}$, with the initial condition $x_0 = \mathcal{S}x$, where the operator $\mathcal{S}$ is given by

$$\mathcal{S}x = \frac{1}{1+r^2} \cdot \frac{\Omega}{\pi} \sum_{k\in\mathbb{Z}} T_k x(s_k) g(t - s_k). \tag{22}$$

The relevance of $\mathcal{S}$ in our context is provided by the following theorem [1]:

Theorem 4 (Reconstruction from Irregular Samples) *If $r = \frac{\delta}{1-c} \cdot \frac{\Omega}{\pi} < 1$ the bandlimited signal x can be perfectly recovered from its samples $(x(s_k))$, $k \in \mathbb{Z}$, as*

$$\lim_{l\to\infty} x_l(t) = x(t), \tag{23}$$

and $\| x - x_l \| \leq (\frac{2r}{1+r^2})^{l+1} \| x \|$.

Proof: See [1], Theorem 6.

Remark 2 A key difference between irregular sampling and time encoding derives from the functional relationship between the trigger times (t_k), $k \in \mathbb{Z}$, and the associated time sequence (s_k), $k \in \mathbb{Z}$, on the one hand and the bandlimited signal on the other. In the case of time encoding, the t_k's are signal dependent. This is clearly underscored by equation (1). For irregular sampling, however, the s_k's are, in general, *signal independent*.

5.2. Upper Bound for the Amplitude Quantization Error

Assume that the instances s_k are exactly known and the amplitudes $x(s_k)$ are corrupted by a sequence of random variables ϵ_k to $x(s_k) + \epsilon_k$.

Proposition 2 *If the random variables (ϵ_k), $k \in \mathbb{Z}$, are independent uniformly distributed within $[-\varepsilon/2, \varepsilon/2]$ then*

$$\mathbb{E}\{\mathcal{E}^2\} \leq \frac{r}{(1-r)^2} \frac{1+c}{1-c} \frac{\varepsilon^2}{12}. \tag{24}$$

Proof: See [4].

Example 3 A reasonable comparison between the effects of amplitude and time quantization can be established if we assume that the quantized amplitudes and quantized trigger times are transmitted at the same bitrate. Since $x(s_k)$ and T_k are associated with the trigger times t_k and t_{k+1}, the same transmission bitrate is achieved if the $x(s_k)$'s and the T_k's are represented by the same number of bits N. With $-c \leq x \leq c$, the amplitude quantization step amounts to $\varepsilon = 2c/2^N$.

For time encoding $T_{\min} = \min_{k\in\mathbb{Z}} T_k \leq T_k \leq \max_{k\in\mathbb{Z}} T_k = T_{\max}$, or equivalently $0 \leq T_k - T_{\min} \leq T_{\max} - T_{\min}$. Therefore, if $T_{\min}$ is exactly known, then only measuring $T_k - T_{\min}$, $k \in \mathbb{Z}$, in the range $(0, T_{\max} - T_{\min})$ is needed. Hence:

$$\Delta = \frac{T_{\max} - T_{\min}}{2^N} = \frac{1}{2^N}\left(\frac{\delta}{1-c} - \frac{\delta}{1+c}\right) = \frac{\delta\varepsilon}{1-c^2}.$$

Substituting the values of ε and Δ above into (20) and (24) results exactly in the same upper bound for both the expected mean square error for time encoding and irregular sampling, respectively.

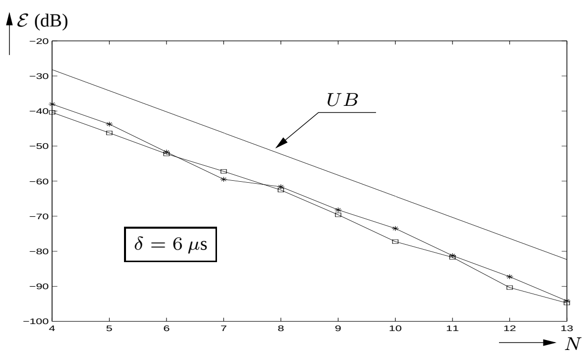

Fig. 4. The dependence of $\mathcal{E}$ on the number of quantization bits for time encoding (stars) and irregular sampling (squares).

Figure 4 shows the mean square error $\mathcal{E}$ as a function of the number of quantization bits, N. The details of the simulation are as before. Squares and stars depict the mean square error for time encoding and irregular sampling, respectively. Figure 4 also depicts the (same) upper bound, UB, arising in inequality (20) and (24).

6. CONCLUSIONS

In this paper we have further established time encoding as an alternative information representation modality for bandlimited signals. We have shown how to construct a TDM that only depends on the time sequence generated by the TEM. No additional knowledge about the parameters of the TEM is required.

We derived an upper bound on the expected mean square error of signal recovery when a quantized version of the trigger times is available. We have also shown that quantization in the time and amplitude domains leads to largely equivalent methods of information representation.

7. REFERENCES

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