On the Robustness of an Analog VLSI Implementation of a Time Encoding Machine

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Abstract—Time encoding is a mechanism for representing the information contained in a continuous time, bandlimited, analog signal as the zero-crossings of a binary signal. Time decoding algorithms have been developed that make a perfect recovery of time encoded bandlimited signals possible. We consider a simple one-opamp active RC implementation of the time encoder and investigate the robustness in performance of the time decoder when the former is subject to non-idealities of the analog VLSI realization. We show that up to a constant scaling factor, delay and offset the input signal can be accurately reconstructed even if the opamp has a finite DC gain and finite bandwidth and the circuit exhibits parameter offsets. We develop an experimental upper bound for the reconstruction error that can be used in the design of the encoder. The analytical results are verified with numerical simulations.

I. INTRODUCTION

The classical sampling theorem calls for reconstructing an analog signal \( x(t) \) bandlimited to \([-\Omega, \Omega]\) using its amplitude samples taken uniformly at or above the Nyquist rate \( \Omega/\pi \). If amplitude samples \( x(t_k) \) are available at irregularly spaced times \( t_k \), the same bandlimited signal \( x(t) \) can be reconstructed if the average of the durations

\[
T_k = t_{k+1} - t_k
\]

are at or below \( \pi/\Omega \) [1]. For signal reconstruction in the irregular sampling case, both the amplitude and time samples are required. In the case of encoding a signal-dependent sampling is carried out by a Time Encoding Machine (TEM) such that \( x(t) \) can be reconstructed based on the \( t_k \)'s only. The TEM investigated in [2] is shown in Figure 1. It consists of an adder, an integrator with unity-gain frequency \( \omega_0 \) and a noninverting Schmitt trigger with a symmetrically centered hysteresis of width \( 2\delta \) and height \( 2b \). In what follows we assume that for some given \( c \)

\[
|x(t)| \leq c < b
\]

holds. To simplify the notation we will also use

\[
\alpha = \frac{c}{b} \quad \text{and} \quad T_0 = \frac{4\delta}{\omega_0 b}
\]

as the bound for the normalized input \( x(t)/b \) and the self-oscillation period of \( z(t) \) for \( x(t) = 0 \), respectively.

![Fig. 1. The ideal time encoder.](image)

Since the integrator's input is \( x(t) \pm b \) where \( |x(t)| < b \) (see (2)), its output \( y(t) \) is a strictly monotonic increasing or decreasing function for \( t \in \{t_k, t_{k+1}\} \) where \( z(t) \) changes sign and \( y(t) \) reaches \( \delta \) or \(-\delta \) at \( t = t_k \). A straightforward analysis [2] gives:

\[
q_k = \int_{t_k}^{t_{k+1}} x(t) dt = (-1)^k b \left( \frac{T_0}{2} - T_k \right)
\]

It can also be shown [2] that \( T_k \) introduced in (1) is bounded as:

\[
\frac{T_0}{2} \frac{1}{1 + \alpha} \leq T_k \leq \frac{T_0}{2} \frac{1}{1 - \alpha}
\]

Original Reconstruction [2]: With \( g(t) = \sin(\Omega t)/(\pi t) \) as the impulse response of an ideal low pass filter of bandwidth \( \Omega \), the reconstructed signal, \( x_r(t) \), is given by

\[
x_r(t) = \sum_{t_k = -\infty}^{\infty} c_k g \left( t - t_k - \frac{T_k}{2} \right),
\]

where the coefficients \( c_k \) are to be found. Integrating both sides of (6) from \( t = t_k \) to \( t = t_{k+1} \) gives \( Gc = q \), where the vectors \( c \) and \( q \) contain the coefficients \( c_k \), and the integrals \( q_k \) in (4), respectively, and the matrix \( G \) is given by

\[
G_{k,\ell} = \int_{t_k}^{t_{k+1}} g \left( t - t_{\ell} - \frac{T_\ell}{2} \right) dt.
\]

One way of solving the typically ill-conditioned linear equations \( Gc = q \) is given by

\[
c = G^+q,
\]

where \( G^+ \) denotes the pseudo (Moore-Penrose) inverse of \( G \). As in the case of irregular sampling, perfect reconstruction is possible if the average density of the \( t_k \)'s is below or at the Nyquist rate [2]. Based on (5) this condition is certainly satisfied if \( (T_0/2)/(1 - \alpha) < \pi/\Omega \).

Compensation Principle [2]: The reconstruction based on (4) relies on the accurate value of \( T_0 \). An alternative technique is based on the simple observation that due to the additive property of integrals \( q_k + q_{k+1} = (-1)^k b(T_{k+1} - T_k) \) does not depend on \( T_0 \). Forming a vector with elements \( q_k + q_{k+1} \) is possible as

\[
\text{vec} \left[ Bq \right] = \int_{t_k}^{t_{k+1}} x(t) dt = (-1)^k b(T_{k+1} - T_k),
\]

where matrix \( B \) is defined as \( B_{k,\ell} = 1 \) for \( k = \ell \) and \( k = \ell + 1 \) and \( B_{k,\ell} = 0 \) otherwise. Using \( B \) and \( Gc = q \) gives \( BGc = Bq \), therefore:

\[
c = (B^+G)q.
\]

Remark 1: Perfect reconstruction can be achieved only if the dimensionality of the matrices and vectors used \((G, B, q, \text{and } c)\) is infinite. Since in reality only a finite number of \( t_k \)'s is available within some observation window \( t_k \in (T_{max}, T_{min}) \), \( x_r(t) \) of (6) merely approximates \( x(t) \). Due to boundary effects [2] the error in signal...
II. ACTIVE-RC TEM

Figure 2 shows an active RC implementation\(^1\) of the encoder, where \(x(t), v(t), y(t),\) and \(z(t)\) denote node voltages. The parameters \(\Delta_0\) and \(\Delta_b\) model the implementation errors in the hysteresis levels \(\pm \Delta_0\) and \(\pm b\). The opamp shown has infinite input and zero output impedance and an input referred offset voltage \(V_{os}\). For \(\Delta_0 = \Delta_b = V_{os} = 0\) the circuit in Fig. 2 implements the ideal TEM of Fig. 1 with \(w_0 = 1/(RC)\).

For the input voltage \(v_d(t) = V_{os} - v(t)\) the linear imperfections\(^2\) of the opamp are modeled by the transfer function

\[
A(s) = -\frac{Y(s)}{V_d(s)} = \frac{A_{DC}}{1 + s\frac{\Delta_b}{\omega_0}},
\]

where \(A_{DC}\) and \(\omega_0\) are the finite DC gain and unity-gain frequency (gain-bandwidth product) [3].

A. Effects of parameter errors and opamp offset

Assuming \(A_{DC} \to \infty\) and \(\omega_0 \to \infty\), we have (see Appendix)

\[
\gamma J \int_{t_k}^{t_{k+2}} (x(t) - X_{os}) dt = b(-1)^k(T_{k+1} - T_k),
\]

where \(\gamma = b / (b + \Delta_b / 2)\) and \(X_{os} = 2V_{os} + \Delta_b / 2\).

Comparing (12) with (9) we conclude that the reconstructed signal using the Compensation Principle is \(x(t)\) except for a constant scaling factor \(\gamma\) and DC offset \(X_{os}\) due to opamp offset, \(V_{os}\), and the hysteresis output level variation \(\Delta_0\). The variation in \(\Delta_0, \Delta_b\), does not affect the encoding as long as the Compensation Principle is used for reconstruction.

B. Effects of the opamp’s finite DC gain and gain-bandwidth product

To simplify the notations we assume that \(\Delta_0 = \Delta_b = 0\), and \(V_{os} = 0\). The opamp’s finite DC gain results in a maximum DC gain for the integrator and the creation of a (parasitic) low frequency pole \(\omega_1\); the finite gain-bandwidth product results in a (parasitic) high frequency pole, \(\omega_2\), and using (11) the transfer function for the integrator becomes [4]

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{-Z(s)} = \frac{\omega_0 \omega_2}{(s + \omega_1)(s + \omega_2)}
\]

1To simplify the diagrams and notation we show a single ended implementation whereas in an actual VLSI realization a differential implementation would be used. The realization of the \(-1\) multipliers then corresponds to a simple crossing of wires. The presented analysis is valid for differential implementations and can also be easily adapted for alternative TEM realizations using, e.g., transconductor-C integrators.

2Referring to the waveforms in Fig. 3 (a) it is clear that opamp can be designed with a sufficient slew rate and output range so that no significant non-linear errors occur.

For large enough \(A_{DC}\) and \(\omega_0\), the approximations\(^3\)

\[\omega_0 \approx \frac{1}{RC}, \quad \omega_1 \approx \frac{2\omega_0}{A_{DC}}\]

\(\omega_2 \approx \omega_0\), can be used. For \(A_{DC} \to \infty\) and \(\omega_0 \to \infty\), \(H(s)\) indeed becomes the ideal integrator transfer function \(\omega_0 / s\).

C. Simulation example using the model in (13)

A simulation model for the active RC TEM was developed with the opamp replaced by an equivalent circuit\(^4\) to model its finite DC gain and gain-bandwidth product and the following TEM parameters: \(1/(RC) = 12\) Mrad/sec, \(b = 1\) V, \(\Delta = 0.5\) V. A two-tone sinusoid, \(x(t) = A_1 \sin(2\pi f_1 t) + A_2 \sin(2\pi f_2 t)\), where \(A_1 = A_2 = 0.25\) V, \(f_1 = 1\) MHz, and \(f_1 = 1.5\) MHz was used as the input signal (see Figure 4(a)) so that \(\Omega = 2\pi \times 1.5\) Mrad/sec and \(\alpha = 1/2\). These parameters are kept fixed for all examples in this paper.

Ideal opamp: Simulations were carried out to determine \(y(t)\) and \(z(t)\) and 50 zero-crossings of \(z(t)\) were calculated with high accuracy. The reconstructed signal, \(x_r(t)\), was calculated using the original and the Compensation Principle based reconstruction. The root-mean-square (RMS) value of the reconstruction error \(e(t) = x_r(t) - x(t)\) is denoted by \(E\). To avoid boundary effects (see Remark 1) \(E\) is calculated over the reduced range \(t \in (t_1, t_2)\). The reconstruction error \(e(t)\) for a TEM with an almost ideal opamp, \(A_{DC} = 10^7, \omega_0/\omega_0 = 3 \times 10^7\), was calculated first. The original reconstruction and Compensation Principle give almost identical errors with \(E \leq -124\) dB. This error is mostly due to the fact that only 50 \(t_k\)’s were used [2].

Non-ideal opamp: Figure 3 shows \(y(t)\) and \(z(t)\) with nearly ideal and very non-ideal settings for the opamp. In the nearly ideal case \(z(t)\) has transitions (zero-crossings) at \(t = t_k\), when \(y(t)\) reaches \(\delta\) or \(-\delta\). For the non-ideal case \(y(t)\) goes noticeably above \(\delta\) and below \(-\delta\) at \(t = t_k\) and the overall operation of the encoder is slower compared to using an (almost) ideal integrator. For the TEM with a non-ideal opamp, the signal was again reconstructed using the original and Compensation Principle. The RMS reconstruction errors, \(E\), are shown in Figure 4(b) for different opamp parameters: \(A_{DC} = 10, 10^2, \ldots, 10^5\) and \(\omega_0/\omega_0 = 3, 30, 300, 3000\). Using the Compensation Principle \(E\) significantly improves compared to the original reconstruction.

III. ANALYSIS OF THE ERRORS DUE TO THE MODEL IN (13)

The effects of finite opamp gain and gain-bandwidth product can be analyzed using Figure 5 which is obtained by simple rearrangement

3In the simulation examples below the exact values for \(\omega_0\) and \(\omega_2\) were used. In the remainder of the text we also use the notation \(\omega_0/\omega_0\) to compare the opamp’s \(\omega_0\) to the integrator’s \(\omega_0\). In the simulations, \(\omega_0/\omega_0\) was set to the values mentioned but the error in this approximation is very small.

4We used a voltage controlled current source loaded with a resistor and capacitor in parallel and followed by an inverting unity-gain buffer.
of Figure 1 after replacing the integrator by $H(x)$ in (13). Using

$$x(t) \rightarrow \omega_2(s + \omega_2) \rightarrow \int_{t_1}^{t_2} x(t) e^{-\omega_2(t_2-t)} dt = \left(-\frac{1}{\omega_2}ight) + 1 \quad (14)$$

$\omega_2 / s + \omega_2$ block in the feedback path. This intuitively explains

Error in the original reconstruction: For small enough $\omega_1$, the left-hand-side (LHS) of (14) is close to $q_2 / b$ in (4). For sufficiently large $\omega_2$ (15) implies $a_t \simeq (-1)^f$ and therefore approximating the right-hand-side (RHS) of (14) for small $\omega_1$ gives

$$q_t \simeq (-1)^f \left( \frac{T_0}{2} + \frac{2}{\omega_2} - T_e \right). \quad (18)$$

By comparing this result with (4) it can be seen that the constant term $T_0 / 2$ is modified by $2 / \omega_2$. This affects the accuracy of the original reconstruction. In agreement with the solid traces of Figure 4(b), increasing $\omega_2$ (hence $\omega_0$) by a factor of 10 results in about 20 dB drop in $\mathcal{E}$ provided that $A_{\text{op}}$ is large enough (hence $\omega_0$ is small enough).

Input delay due to $\omega_2$: Since $\tilde{x}(t)$, introduced in Fig. 5 is merely a slightly pre-filtered version of $x(t)$, a reconstruction for $\tilde{x}(t)$ might be acceptable in many applications. Moreover, for reasonably large $\omega_2$ (see the Appendix)

$$\tilde{x}(t) \simeq x (t - \frac{1}{\omega_2}) \quad (19)$$

and thus $\tilde{x}(t)$ is just a delayed version of the input. Simulation results: Here we use the same parameters and input as in Section II-C. The reconstructed signal, $x_r(t)$ is calculated using the Compensation Principle; $\mathcal{E}$ denotes the RMS value of the error signal $\tilde{e}(t) = x(t) - x_r(t)$; $\mathcal{E}_d$ is the RMS value of the error with respect to the delayed input, $e_d(t) = x(t - 1 / \omega_2) - x_r(t)$. Figure 6(a) shows the simulation results for $\mathcal{E}_d$. A very substantial improvement of 20 dB or better can be seen compared to the dashed traces (Compensation Principle) of Figure 4(b). As expected from the presented analysis, an important effect of the presence of $\omega_2$ is the creation of a delay between the input signal $x(t)$ and the encoded signal in $z(t)$. In most applications this delay is acceptable and thus the results in Figure 6(a) depict the precision that can be attained with different opamp realizations of the TEM.

Comparing $\mathcal{E}_d$ and $\mathcal{E}$ in Figure 6(a) we conclude that for $\omega_0 / \omega_0 = 3$ or 30 the approximation of the effect of $\omega_2$ as a simple delay in (19) is not accurate. However, $x(t)$ now encodes $\tilde{x}(t)$ very accurately even for a low $\omega_0 / \omega_0$ of 30.

Output delays in the hysteresis: The finite output bandwidth of the hysteresis circuit results in a low pass filter in the feedback path for $z(t)$ and its model corresponds to Figure 5 but without the input $\omega_2 / s + \omega_2$ block. Its effects can be quantified as the effects of $\omega_2$ on the reconstruction error for $\tilde{x}(t)$. Provided that the output bandwidth of the hysteresis circuit is larger than the unity gain bandwidth of the opamp, the former will not significantly affect the TEM accuracy.

Error bound for the integrals and estimate for the RMS reconstruction error: The recovery error is intuitively expected to decrease if the reconstructed signal is compared to $\tilde{x}(t)$ of Figure 5, since the undesired prefiltering of the $\omega_2 / s + \omega_2$ block is ignored. In particular, as shown in Appendix we have the approximation

$$\int_{t_k}^{t_{k+2}} \tilde{x}(t) dt \simeq b(-1)^f (T_{k+1} - T_k) + e_k, \quad (20)$$

where the integral error sequence $e_k$ is bounded as:

$$|e_k| \leq c \left[ c \omega_2 (4 + \omega_2) |b| (6 - 2 \alpha + \omega_2 T_0) \right] \frac{2(\alpha - 1)^2 (\alpha + 1) \omega_2^2}{2(\alpha - 1)^2 (\alpha + 1) \omega_2^2}. \quad (21)$$

Based on the integral error $e_k$ in (20) it is not straightforward to handle the overall reconstruction error or its RMS value $\mathcal{E}_R$ analytically because of the irregular nature of the $t_k$'s. Nevertheless, a proportionality between $\mathcal{E}$ and $\mathcal{E}_R$ certainly holds due to (10) and (6). In [2] an experimental estimate for the RMS value of the reconstruction error due to time quantization was found by approximating (modeling) $\delta$ the error sequence as i.i.d random variables and ii) the $T_k$'s by their average value $T_e$. Following a similar line of reasoning here, $\mathcal{E}_R$ can be estimated by $\mathcal{E} \sqrt{\Omega / (2 \pi T_e)}$ where, based on (17), a reasonable estimate for $T_e$ is:

$$T_e = \frac{1}{2} \left\{ \left( \frac{T_0}{2} + \frac{2}{\omega_2} \right) \frac{1}{1 - \alpha} \right\} \left\{ \left( \frac{T_0}{2} + \frac{2}{\omega_2} \right) \frac{1}{1 + \alpha} \right\}. \quad (22)$$

Therefore, using (21) a rough estimate for the RMS value of the reconstruction error bound is given by:

$$\mathcal{E}_B = c \sqrt{\Omega / \pi T_e}. \quad (23)$$
Figure 6(b) compares the bound estimates $\mathcal{E}_b$ and the results for $\tilde{\mathcal{E}}$. For $\omega_0/\omega_a = 3$, $\mathcal{E}_b$ reaches $\tilde{\mathcal{E}}$ which is to be expected since third condition in (16) is not sufficiently satisfied ($\omega_0 \text{min}(T_1) = 7.19$ for $A_2 = 10^5$). For larger $\omega_0/\omega_a$, $\omega_0/\omega_a > 3$, $\mathcal{E}_b$ tracks $\tilde{\mathcal{E}}$ but $\mathcal{E}_b$ is not a tight bound. This could be due to finite precision in the simulations or the approximations made in the derivation of $\mathcal{E}_b$. Most probably the main reason is the fact that after canceling a dominant common error term with the Compensation Principle, the estimation of the remaining error is hard.

![Plot](image)

(a) $\mathrm{lg}(\mathcal{E}_b)$, (b) $\mathrm{lg}(\mathcal{E}_b)$.

**IV. DISCUSSION**

The analysis and simulation results presented in this paper show that the TEM operation is robust with respect to VLSI implementation non-idealities. Offsets and some variations in the building block parameters only lead to offsets and scaling factor in the encoded signal. Other parameter variations are completely canceled. The requirements on the integrator circuit can be derived from Figure 6(a). For example, with an opamp DC gain of 60 dB and gain-bandwidth product $\omega_0$ of 30 times the integrator unity gain frequency $f_{\mu}$, the RMS errors remains below -80 dB. For a DC gain of 80 dB and $\omega_0$ = 300$\mu$Hz, the errors are below -110 dB. Such opamp requirements are similar to, e.g., the requirements for opamps for continuous time filters [4] or analog to digital converters [3]. Due to space limitations, we could not address here the robustness properties of the time decoder with respect to thermal noise disturbances arising in the time encoder. These and related questions will be addressed elsewhere.

**APPENDIX**

**Proof of (12):** For an ideal opamp $v(t) = V_o$, and thus the encoder in Figure 2 is described by the circuit equation $(x - V_{oa})/R - (z - V_{oa})/R + C_d(-y - V_{oa})/dt = (x - V_{oa})/R - (z - V_{oa})/R - C_dy/dt = 0$. By integrating both sides above from $t_i$ to $t_{i+1}$ we obtain:

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} \frac{x(t) - V_{oa}}{R} dt - \int_{t_i}^{t_{i+1}} \frac{z(t) + V_{oa}}{R} dt$$

Adding this relationship for $\ell = k$ and $\ell = k-1$ and noting that (without loss of generality) $y(t_i) = -(-1)^{\delta} + (1 - (-1)^{\delta}) T_2/2$ and $z(t) = -(-1)^{\delta} + (1 - (-1)^{\delta}) T_2/2$ for $t \in (t_i,t_{i+1})$ gives (12).

**Proof of (14) and (15):** In terms of the $\omega_0/(s + \omega_0)$ block and the $\omega_0/(s + \omega_2)$ block in the feedback path of Figure 5 we have:

$$y(t) = e^{-\omega_0(t-t_i)}y(t_i) + \omega_0 \int_{t_i}^{t} (\tilde{z}(t) - z(t)) e^{-\omega_0(t-t)} dt$$

and

$$\tilde{z}(t) = e^{-\omega_0(t-t_i)}z(t_i) + \omega_0 \int_{t_i}^{t} z(t) e^{-\omega_0(t-t)} dt,$$

respectively, for some $t \in (t_i,t_{i+1})$. Due to the presence of the Schmitt trigger, $y(t_i) = -(-1)^{\delta}$ and $z(t) = -(-1)^{\delta} b$. Using these relationships with $a_t = \xi(t)/b$ and $T_0$ defined in (3) in evaluating (24) and (25) at $t = t_{i+1}$ gives (14) and (15).

**Proof of (17):** Expressing $\tilde{x}(t)$ as a convolution and using (2) gives:

$$\tilde{x}(t) = \int_{t_0}^{t} x(t-\tau) \omega_2 e^{-\omega_0 \tau} d\tau \leq c \int_{t_0}^{t} \omega_2 e^{-\omega_0 \tau} d\tau = c.$$  (26)

Using the third condition of (16) in (15) gives $a_{m+1} \approx -(-1)^{\ell}$. Therefore the RHS of (14) can be approximated by $(-1)^{\ell}(T_0/2 + 2/\omega_0 - T_1)$. By applying the mean value theorem the integral on the LHS of (14) can be expressed as $T_0^e(\xi/\omega_2) e^{-\omega_0(t_{i+1}-\xi)} \approx T_0^c \xi/\omega_2$ with appropriate $\xi/\omega_2 \in [t_i,t_{i+1}]$. Combining these relationships gives the inequality $(-1)^{\ell}T_0^c/b > 2T_0/2 - 2/\omega_0 - T_1$ that can be solved for $T_1$ (depending on $\ell$ being even or odd) and (17) follows.

**Proof of (19):** Let $X(\omega)$ be the Fourier transform of $x(t)$. Then, using the transfer function $\omega_2/(j\omega + \omega_2) = \omega_2 e^{-j\text{arg} \omega_2/2}/\sqrt{\omega^2 + \omega_2^2}$ corresponding to the $\omega_2/(s + \omega_2)$ block in Fig. 5 and the inverse Fourier transform representation gives:

$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \frac{\omega_2}{\sqrt{\omega^2 + \omega_2^2}} e^{j\omega - j\text{arg} \omega_2/2} d\omega.$$  (26)

With the first assumption in (16) and noting that the integration range is carried out over $\omega \in (-\Omega, \Omega)$ we obtain $\omega_2/\sqrt{\omega^2 + \omega_2^2} \approx 1$ and $\text{arg} \omega_2/\omega_2 \approx \omega_2/\omega_2$. Therefore

$$\tilde{x}(t) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega} d\omega = x(t - \frac{1}{\omega_0})$$

in agreement with (19).

**Proof of (20) and (21):** Rearranging (14) and approximating by using (16) and (26) gives:

$$\int_{t_i}^{t_{i+1}} \tilde{x}(t) \approx \int_{t_i}^{t_{i+1}} \tilde{x}(t) \left[1 - e^{-\omega_0(t_{i+1} - t)}\right] + 2b(-1)^{k} T_0^c/\omega_2 + b(-1)^{k} T_0/4 \left(-\omega_0 T_0 + 2\right).$$

Adding this relationship for $\ell = k$ and $\ell = k + 1$ gives (20) with

$$e_k \leq \frac{c}{2} \omega_0(T_k^2 + T_k^1) + (-1)^k \omega_0(T_k + 1 - A_2) \left(\frac{2}{\omega_0} + T_0^2/4\right).$$

From (17) we obtain $T_k^2 + T_k^1 \leq 2(T_0/2 + 2/\omega_2)/(1 - \alpha^2)$ and $T_k^0 + T_k^1 \leq (T_0/2 + 2/\omega_2)(1/(1 - \alpha) - 1/(1 + \alpha))$ and the bound in (21) follows.

**REFERENCES**


