Encoding of Multivariate Stimuli with MIMO Neural Circuits

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***Abstract*—We present a general MIMO neural circuit architecture for the encoding of multivariate stimuli in the time domain. The signals belong to the finite space of vector-valued trigonometric polynomials. They are filtered with a linear time-invariant kernel and then processed by a population of leaky integrate-and-fire neurons. We present formal, intuitive, necessary conditions for faithful encoding and provide a perfect recovery (decoding) algorithm. We extend these results to multivariate product spaces and apply them to video encoding with MIMO neural circuits. We demonstrate that our encoding circuits can serve as measurement devices for compressed sensing of frequency sparse signals. Finally, we provide necessary spike density conditions for the decoding of infinite-dimensional vector valued bandlimited functions encoded with MIMO neural circuits.**

***Index Terms*—time encoding, spiking neurons, MIMO sampling, video encoding, compressed sensing.**

I. Introduction

The wide availability of multi-electrode recordings as well as functional imaging techniques that operate at the cellular level has shifted the focus towards population-centric approaches to neural encoding. Multi-input multi-output (MIMO) time encoding machines (TEMs) encode vector-valued analog stimuli into a population of spike trains. Examples of MIMO models abound in the system neuroscience literature (e.g., the antennal lobe in insects). Such models have also been used in brain-machine interfaces [1] as well as silicon retinas and related hardware applications [2].

In this paper we investigate conditions for the faithful representation of analog video with MIMO TEMs. The canonical input stimulus is an M-dimensional vector-valued trigonometric polynomial. Such stimuli are a natural discretization of bandlimited functions in the frequency domain. The stimulus is passed through a linear filter (kernel) whose output is fed to a population of N spiking neurons. For simplicity we consider here neurons of leaky integrate-and-fire (LIF) type. Faithful encoding with spiking neurons has a simple geometric interpretation. The circuit projects the input stimulus onto spike dependent functions that span the space of input signals.

We reduce the recovery problem to the solution of a system of linear equations and derive necessary conditions on the number of spikes and the structure of the filtering kernel that are required for the existence of a solution. Since the canonical input space is finite dimensional, these conditions can be reduced to certain matrix rank conditions that are analytically tractable. We postulate that these necessary conditions are also sufficient. MIMO TEMs were first investigated in [3]; however, the problem of perfect reconstruction was not addressed.

We extend our results to multivariate spaces that are constructed by the tensor product between the space of trigonometric polynomials and an arbitrary finite Hilbert space. We focus on the space of space-time video signals and show that MIMO TEMs can be used for the encoding of video signals within a neural modeling framework. We present several examples for vector-valued and video signals. By using the standard l_1-norm relaxation algorithms from compressed sensing, we also demonstrate that MIMO TEMs can be used for sub-Nyquist sampling of frequency sparse video signals. These insights demonstrate that MIMO TEMs provide a sensing mechanism suitable for a wide range of applications while at the same time leverage the advantages of an asynchronous, temporal code. Finally, we extend our results to multivariate stimuli defined with infinite-dimensional bandlimited temporal components and provide conditions for MIMO generalized sampling.

II. MIMO Neural Population Encoding

In this section we introduce the space of stimuli and describe their representation by the two stages of MIMO TEM architecture: the filtering kernel and the ensemble of spiking neurons.

A. The Space of Trigonometric Polynomials

The stimuli of interest are assumed to belong to the space of vector-valued trigonometric polynomials. The latter consists of functions that are simultaneously bandlimited with bandwidth Ω (in rad/sec) and periodic with period $T = 2\pi S/\Omega$, where S is a positive integer that denotes the order (resolution) of the space. An element $\mathbf{u}$ of the space (denoted by $\mathcal{H}_S^M$) is of the form $\mathbf{u} = [u^1, \ldots, u^M]'$, with

$$u^i(t) = \frac{1}{\sqrt{T}} \sum_{s=-S}^{S} a_s^i \exp\left(js\omega_S t\right), \quad t \in [-T/2, T/2], \quad (1)$$

where $\omega_S = \Omega/S$. The space $\mathcal{H}_S^1$ is a natural discretization of the space of bandlimited functions in the frequency domain. The exponentials in (1) have a line Fourier spectrum at the points $s\omega_S$ with $s = -S, \ldots, S$. By letting $S \to \infty$, this spectrum becomes dense in $[-\Omega, \Omega]$. We assume, wlog, that

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all the input components have the same bandwidth and the same resolution. Since all stimuli are real signals, $a^i_{-m} = \overline{a^i_m}$. The inner product on $\mathcal{H}_S^M$ is given by

$$\langle \mathbf{u}, \mathbf{w} \rangle = \frac{1}{M} \sum_{i=1}^{M} \int_{-T/2}^{T/2} u^i(s)\overline{w^i(s)}\, ds. \tag{2}$$

Under (2), the set of functions $(\mathbf{e}^i_s), i = 1, \ldots, M, s = -S, \ldots, S$, whose i-th component is equal to $\exp\left(js\omega_S t\right)/\sqrt{T}$ and zero, otherwise, constitutes an orthonormal basis (ONB) for $\mathcal{H}_S^M$.

B. Stimulus Encoding with the Filtering Kernel

Stimuli in $\mathcal{H}_S^M$ are encoded with a MIMO TEM architecture consisting of an $N \times M$-dimensional filtering kernel and an ensemble of N neural circuits. The latter, in their most general form, are characterized by piecewise linear dynamics and spike-triggered feedback (pulse coupling) [3]. Here, for simplicity we assume that each neural circuit consists of a single LIF neuron.

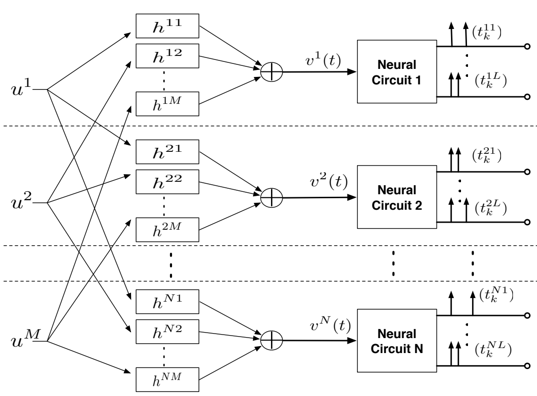

Fig. 1. Multi-Input Multi-Output Time Encoding Machine.

Let $\mathbf{H} : [-T/2, T/2] \mapsto \mathbb{R}^{N\times M}$ be a filtering kernel defined as a $N \times M$ matrix-valued function with $[\mathbf{H}(t)]_{ji} = h^{ji}(t)$. We assume that any filter $h^{ji}, j = 1, \ldots, N, i = 1, \ldots, M$, of the kernel belongs to $\mathcal{H}_S^1$ and can be written in the form of (1) with coefficients $h^{ji}_s, s = -S, \ldots, S$. Each filter of the kernel receives input from one of the M component inputs and its output is additively coupled into a single neuron (see Fig. 1). Filtering $\mathbf{u}$ with $\mathbf{H}$ leads to a signal $\mathbf{v} = [v^1, v^2, \ldots, v^N]'$ in $\mathcal{H}_S^N$ where

$$\begin{aligned} v^j(t) &= \sum_{i=1}^{M} (h^{ji} * u^i)(t) = \sum_{s=-S}^{S} v^j_s e_s(t), \text{ with} \\ v^j_s &= \sqrt{T} \sum_{i=1}^{M} a^i_s h^{ji}_s. \end{aligned} \tag{3}$$

For each frequency $s\Omega/S, s = -S, \ldots, S$, (3) can be written

$$\mathbf{H}_s \mathbf{a}_s = \mathbf{v}_s, \tag{4}$$

where $\mathbf{H}_s$ is a $N \times M$ matrix with $[\mathbf{H}_s]_{ji} = h^{ji}_s$, $\mathbf{a}_s = [a^1_s, a^2_s, \ldots, a^M_s]'$ and $\mathbf{v}_s = [v^1_s, v^2_s, \ldots, v^N_s]'$. Regardless of the spiking behavior of the neural circuits, a trivial necessary condition for stimulus recovery, is that (4) has a solution for every s. Equivalently, $\mathbf{H}_s$ has rank M. A necessary condition for the latter is $N \geq M$, i.e., the number of neurons is greater or equal to the number of inputs.

C. Stimulus Encoding with the Integrate-and-Fire Neurons

The neural model that we employ here is the LIF neuron. The stimulus v biased by a constant background current b is fed into a LIF neuron with threshold δ, resistance R and capacitance C. Assume that after each spike the neuron is reset to zero. The operation of the neuron is described by the piecewise linear differential equation

$$C\tfrac{dV(t)}{dt} = -\tfrac{V(t)}{R} + v(t) + b, \text{ with } V(t_k) = \delta \Rightarrow \lim_{t\to t_k^+} V(t) = 0.$$

Let $(t_k), k = 1, 2, \ldots, n{+}1$, denote the output spike train of the LIF neuron. By solving the equation between two consecutive spike times, we obtain the t-transform equations

$$\int_{t_k}^{t_{k+1}} \exp\left(-\tfrac{t_{k+1}-t}{RC}\right)(b + v(t))\, dt = C\delta. \tag{5}$$

The measurements of (5) can be written in the inner product form

$$\langle v, \chi_k \rangle = q_k, \quad q_k = C\delta - bRC\left(1 - \exp\left(-\tfrac{t_{k+1}-t_k}{RC}\right)\right) \tag{6}$$

and the sampling functions $\chi_k, k = 1, 2, \ldots, n$, can be expressed as $\chi_k = \sum_{|s|\leq S} b_{s,k} e_s$, where the coefficients $b_{s,k}$ are given by

$$b_{s,k} = \frac{1}{\sqrt{T}} \int_{t_k}^{t_{k+1}} e^{-\frac{t_{k+1}-t}{RC}}\, \overline{e_s}(t)\, dt. \tag{7}$$

Lemma 1. *Let $\mathbf{B} \in \mathbb{R}^{n\times(2S+1)}$ with $[\mathbf{B}]_{k,s} = b_{s-S-1,k}$. Then*

$$r(\mathbf{B}) = \min(n, 2S + 1).$$

Proof: The matrix $\mathbf{B}$ can be written as a product of an upper-triangular, a Vandermonde and a diagonal matrix, all of which are of full rank. ∎

III. Stimulus Reconstruction

In this section we discuss the problem of perfect recovery of the input vector-valued stimulus from the set of spike times. We provide necessary conditions regarding the spiking density of the neurons and the frequency support of the filtering matrix $\mathbf{H}$. Finally we present an algorithm that can perfectly recover the stimulus.

A. Necessary Conditions for Perfect Recovery

We assume that every LIF neuron j fired a total of $n_j + 1$ spikes, $j = 1, 2, \ldots, N$. Using (3), the t-transform equation $\langle v^j, \chi^j_k \rangle = q^j_k$ can be rewritten as

$$\sum_s \left(\sum_{i=1}^{M} \sqrt{T} a^i_s h^{ji}_s\right) \overline{b^j_{s,k}} = q^j_k \Rightarrow \sum_{i=1}^{M} \sum_s \sqrt{T}\left(h^{ji}_s \overline{b^j_{s,k}}\right) a^i_s = q^j_k. \tag{8}$$

Writing (8) for all $k = 1, 2, \ldots, n_j$, we obtain in matrix form

$$\overline{\mathbf{B}^j}\tilde{\mathbf{H}}^j\mathbf{a} = \mathbf{q}^j, \tag{9}$$

with $\mathbf{q}^j = [q_1^j, \ldots, q_{n_j}^j]'$, $\mathbf{a} = [\mathbf{a}_{-S}; \mathbf{a}_{-S+1}; \ldots; \mathbf{a}_S]$. The matrices $\mathbf{B}^j$ and $\tilde{\mathbf{H}}^j$ have dimensions $n_j \times (2S+1)$ and $(2S+1) \times M(2S+1)$, respectively, and are given by

$$[\mathbf{B}^j]_{kl} = \sqrt{T} b_{l-S-1,k}^j, \quad \tilde{\mathbf{H}}^j = \mathrm{diag}(\mathbf{h}_{-S}^j, \ldots, \mathbf{h}_S^j),$$

with $\mathbf{h}_s^j = [h_s^{j1}, h_s^{j2}, \ldots, h_s^{jM}]$. Repeating for all neurons we obtain the system of equations

$$\begin{bmatrix} \overline{\mathbf{B^1}} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{B^2}} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \overline{\mathbf{B^N}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{H}}^\mathbf{1} \\ \tilde{\mathbf{H}}^\mathbf{2} \\ \vdots \\ \tilde{\mathbf{H}}^\mathbf{N} \end{bmatrix} \mathbf{a} = \begin{bmatrix} \mathbf{q^1} \\ \mathbf{q^2} \\ \vdots \\ \mathbf{q^N} \end{bmatrix}, \text{ or } \mathbf{Fa} = \mathbf{q} \tag{10}$$

with $\mathbf{F} = \overline{\mathbf{B}}\tilde{\mathbf{H}}$. In order to recover the vector $\mathbf{a}$, the matrix $\mathbf{F}$ has to be of rank equal to the dimension of $\mathbf{a}$, i.e., $M(2S+1)$.

Lemma 2. *The matrices* $\mathbf{B}$ *and* $\tilde{\mathbf{H}}$ *have rank, respectively,*

$$r(\mathbf{B}) = \sum_{j=1}^{N} \min(n_j, 2S+1), \quad r(\tilde{\mathbf{H}}) = \sum_{s=-S}^{S} r(\mathbf{H}_s). \tag{11}$$

Proof: Since $\mathbf{B}$ is block-diagonal its rank equals the sum of the rank of its blocks and the result follows from Lemma 1. By rearranging the rows of $\tilde{\mathbf{H}}$, we obtain the block-diagonal matrix $\mathbf{H}^* = \mathrm{diag}(\mathbf{H}_{-S}, \mathbf{H}_{-S+1}, \ldots, \mathbf{H}_S)$ and the result follows. ■

Theorem 1. *A necessary condition for perfect recovery is*

$$\min\left(\sum_{j=1}^{N} \min(n_j, r(\tilde{\mathbf{H}}^\mathbf{j})), \sum_{s=-S}^{S} r(\mathbf{H}_s)\right) = M(2S+1) \tag{12}$$

Proof: Since $\mathbf{F} = \overline{\mathbf{B}}\tilde{\mathbf{H}}$, we have that $r(\mathbf{F}) \leq \min(r(\mathbf{B}), r(\tilde{\mathbf{H}}))$ [4]. By applying Lemma 2 we obtain

$$r(\mathbf{F}) \leq \min\left(\sum_{j=1}^{N} \min(n_j, 2S+1), \sum_{s=-S}^{S} r(\mathbf{H}_s)\right). \tag{13}$$

Moreover, by decomposing $\mathbf{F}$ into its blocks we have

$$r(\mathbf{F}) \leq \sum_{j=1}^{N} r(\mathbf{B^j}\tilde{\mathbf{H}}^\mathbf{j}) \leq \sum_{j=1}^{N} \min\left(n_j, 2S+1, r(\tilde{\mathbf{H}}^\mathbf{j})\right). \tag{14}$$

Combining (13) and (14) and noting that $r(\tilde{\mathbf{H}}^\mathbf{j}) \leq 2S+1$ for all $j = 1, 2, \ldots, N$ we get

$$r(\mathbf{F}) \leq \min\left(\sum_{j=1}^{N} \min(n_j, r(\tilde{\mathbf{H}}^\mathbf{j})), \sum_{s=-S}^{S} r(\mathbf{H}_s)\right).$$

(12) follows by noting that the recovery condition is $r(\mathbf{F}) = M(2S+1)$ and $r(\mathbf{H}_s) \leq M$ for all $s = -S, \ldots, S$. ■

Remark 1. *The intuitive full rank condition* $r(\mathbf{H}_s) = M$ *for every frequency* s *is naturally embedded into* (12)*. Moreover,* (12) *shows that the number of linear independent measurements that each neuron contributes is equal to the number of frequency components that its filtering vector supports. The condition substantially improves upon the ones presented in [5], where separated loose conditions were presented for the filtering kernel and the spiking densities.*

Remark 2. *Condition* (12) *is also sufficient if* (14) *holds with strict equality. In practice this always holds as long as the matrices* $\tilde{\mathbf{H}}^\mathbf{j}$ *and the LIF neurons satisfy some 'linear independence condition'. The exact conditions will be pursued elsewhere.*

B. Perfect Recovery Algorithm

To solve the system of complex equations (10) we first reduce it to a system of real equations. This can be achieved by noting that $\mathbf{a}$ and $\mathbf{F}$ have a special structure and can be written in the following form

$$\begin{aligned} \mathbf{F} &= [\overline{\mathbf{F}_S}, \overline{\mathbf{F}_{S-1}}, \ldots, \mathbf{F}_0, \ldots, \mathbf{F}_{S-1}, \mathbf{F}_S] \\ \mathbf{a} &= [\overline{\mathbf{a}_S}; \overline{\mathbf{a}_{S-1}}; \ldots; \mathbf{a}_0; \ldots; \mathbf{a}_{S-1}; \mathbf{a}_S], \end{aligned} \tag{15}$$

where each submatrix $\mathbf{F}_s$ has dimensions $\sum_j n_j \times M$ and each sub-vector $\mathbf{a}_s$ has dimensions $M \times 1$. With that in mind, (10) can be written in the form $\mathbf{F^r a^r} = \mathbf{q}$ with

$$\begin{aligned} \mathbf{F^r} &= [\mathbf{F}_0, 2\Re(\mathbf{F}_1), \ldots, 2\Re(\mathbf{F}_S), -2\Im(\mathbf{F}_1), \ldots, -2\Im(\mathbf{F}_S)] \\ \mathbf{a^r} &= [\mathbf{a}_0; \Re(\mathbf{a}_1); \ldots; \Re(\mathbf{a}_S); \Im(\mathbf{a}_1); \ldots; \Im(\mathbf{a}_S)], \end{aligned}$$

where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary part. We now present an algorithm that can perfectly recover the stimulus.

Algorithm 1. *If* $r(\mathbf{F}) = M(2S+1)$*, then* $\mathbf{u}$ *can be recovered as*

$$\begin{aligned} \mathbf{u} &= \sum_{s=-S}^{S} \left[c_s^1, c_s^2, \ldots, c_s^M\right]' e_s(t) \\ [c_s^1, c_s^2, \ldots, c_s^M] &= \Re(\mathbf{a}_{|s|}) + \mathit{sign}(s)\Im(\mathbf{a}_{|s|}), \end{aligned} \tag{16}$$

with the entries of the vector $\mathbf{a^r}$ *given by* $\mathbf{a^r} = \mathbf{F^{r+}}\mathbf{q}$*, where* $\mathbf{F^{r+}}$ *denotes the pseudoinverse of the matrix* $\mathbf{F^r}$*.*

C. Two Examples

1) Example: Delay Filter Bank: We present the realization of the recovery algorithm for a filtering kernel that induces *arbitrary, but known, delays and weights* on the stimulus. The kernel models dendritic tree latencies in sensory neurons (motor, olfactory) or, in general, delays and synaptic weights between groups of pre- and postsynaptic neurons. Each filter h^{ji} is the form $h^{ji}(t) = w^{ji}\delta(t - d^{ji})$ with $d^{ji} \geq 0$, for all $j = 1, 2\ldots, N$, and all $i = 1, 2, \ldots, M$. When projected onto $\mathcal{H}_S^M$ we obtain $h^{ji}(t) = \sum_{s=-S}^{S} w^{ji} e^{-js\omega_S d^{ji}} e_s(t)$.

The vector valued signal $\mathbf{u}(t)$ was chosen to have four trigonometric components ($M = 4$) all with $\Omega = 2\pi \cdot 80$ Hz and $S = 20$ defined on the interval $[0, 0.25]$sec. In total, 16 ideal IAF neurons were used to encode the signal ($N = 16$). The delays were drawn from an exponential distribution with mean $\pi/3\Omega$. The weights, biases and thresholds were drawn from uniform distributions on the intervals $[0.5, 1]$, $[1.3, 2.3]$

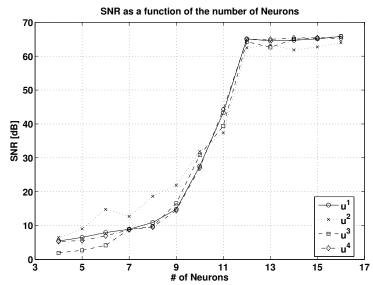

Fig. 2. SNR as a function of the number of neurons.

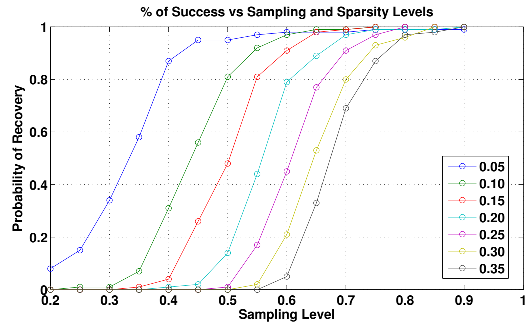

Fig. 3. Probability of successful recovery of sparse, undersampled stimuli as a function of the normalized number of spikes and the level of stimulus sparsity. Each curve corresponds to a different level of sparsity.

and $[1.4, 2.4]$, respectively. Finally, $C^j = 0.01$ for all neurons. The 16 neurons respectively produced 13, 18, 29, 31, 14, 13, 17, 25, 13, 27, 20, 15, 20, 23, 13 and 18 spikes.

Fig. 2 shows the SNR of the recovery of each stimulus component when $4, 5, \ldots, 16$ neurons are used. Overall, as more neurons are added, the SNR increases, eventually reaching very high values (> 60 [dB]). This occurs for the first 12 neurons that have a total of 203 samples, i.e., have a total number of samples above the perfect recovery bound of $M(2S+1) = 163$ spikes.

2) Applications to Compressed Sensing: MIMO TEMs (and more generally TEMs) can also be used as measurement devices for sub-Nyquist sampling of sparse signals and related compressed sensing applications [6], [7]. In our setup a sparse vector-valued stimulus in $\mathcal{H}_S^M$ corresponds to sparsity in the frequency domain. We investigated whether such sparse stimuli can be recovered from the spike times even when the necessary rank conditions are not satisfied by solving the standard l_1-norm optimization problem

$$\mathbf{a}^* = \underset{\mathbf{F^r a^r = q}}{\operatorname{argmin}} \|\mathbf{a^r}\|_{l_1}. \tag{17}$$

We considered an example with a vector-valued stimulus with 4 components ($M = 4$) and $S = 30$, $\Omega = 2\pi \cdot 30$. Several vector-valued stimuli were constructed with levels of sparsity that ranged from 5% up to 35%. 6 neurons were used to encode the stimulus. The filters of the filtering kernel were random with coefficients drawn from a standard complex normal distribution. The neurons were of LIF type and their parameters were chosen in such a way that the total number of spikes approximately ranged from 20% up to 90% of the required rank $M(2S+1)$ given by Theorem 1. The recovery results were obtained with the l_1-magic toolbox [8]. For each combination of sparsity level and total number of spikes 100 repetitions were run with different stimuli and filtering kernels. Fig. 3 shows the probability of perfect recovery (defined as reconstruction with SNR > 40 [dB]) as a function of the number of spikes normalized by the required rank (Nyquist rate equivalent), plotted separately for each sparsity level.

The example shows that the solution of (17) achieves perfect recovery for each sparsity level checked, provided that the number of total spikes is sufficiently high (although that is always below the dimensionality of the system $M(2S+1)$). The example suggests that, in addition to generalized sampling under Nyquist-type conditions, TEMs are a natural real-time measurement (sampling) device suitable for compressed sensing and analog-to-information conversion [9].

IV. Encoding Video with MIMO Neural Circuits

The results of Theorems 1 and 2 (see Appendix A) can be easily extended to the case of video TEMs [10]. Let $(f_i(x,y)), (x,y) \in \mathbb{R}^2, i = 1, \ldots, M$, be an ONB for an image space $\mathcal{V}$. Then the set of functions $(e_s f_i), s = -S, \ldots, S, i = 1, \ldots, M$, is an ONB for the product video space $\mathcal{V} \otimes \mathcal{H}_S$. An element $I \in \mathcal{V} \otimes \mathcal{H}_S$ can be written as

$$I(x,y,t) = \sum_{i=1}^{M} \sum_{s=-S}^{S} a_s^i e_s(t) f_i(x,y) = \sum_{i=1}^{M} u^i(t) f_i(x,y),$$

(with $a_{-s}^i = \overline{a_s^i}$), where $u^i \in \mathcal{H}_S$ and is given by

$$u^i(t) = \sum_{s=-S}^{S} a_s^i e_s(t).$$

Similarly, the spatio-temporal receptive field (STRF) of the neuron $j, j = 1, \ldots, N$, can be written as

$$D^j(x,y,t) = \sum_{i=1}^{M} \sum_{s=-S}^{S} h_s^{ji} e_s(t) f_i(x,y) = \sum_{i=1}^{M} h^{ji}(t) f_i(x,y),$$

with $h^{ji}(t)$ defined similarly (again $h_{-s}^{ji} = \overline{h_s^{ji}}$). In the case of video TEMs (and visual neurons), the filtering is performed by multiplication in the spatial domain and convolution in the time domain. Therefore the output v^j of the STRF of the j-th neuron is a temporal signal given by

$$v^j(t) = \sum_{i=1}^{M} \sum_{m=1}^{M} \langle f_i, f_m \rangle_{\mathcal{V}} (h^{ji} * u^m)(t) = \sum_{i=1}^{M} (h^{ji} * u^i)(t),$$

that has exactly the same form as (3). Therefore, Theorem 1 can be directly applied to video TEMs and, in general, TEMs encoding multivariate signals. Note that this result generalizes the ones presented in [10], since it allows neurons to fire an arbitrary number of spikes.

As an example we demonstrate, how a video stimulus can be faithfully represented in a compressed form by a MIMO TEM

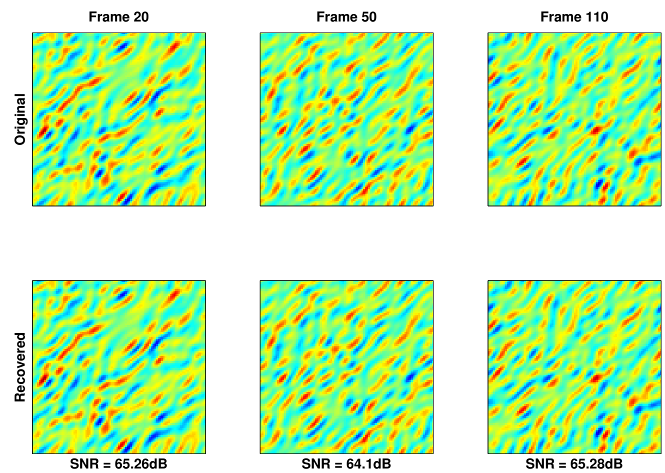

Fig. 4. Perfect reconstruction from compressed sensing, using video TEMs.

in the spike domain. For the image space we used again the space of trigonometric polynomials with resolution (defined in a similar way as in the pure temporal case) $S_x = S_y = 7$. For the temporal domain we had $S = 11$ (total dimensionality 5,175). The input video used was constructed by randomly picking 938 coefficients and assigning them a nonzero value (ending in 18% sparse signal in the frequency domain). The signal was sensed from 600 LIF neurons with appropriate parameters that produced a total of 3,362 spikes, significantly below the total dimensionality of the stimulus. Using the same L_1 minimization algorithm, we are able to reconstruct the video stimulus with a total SNR= 64.97 [dB]. Three representative frames of the reconstruction are shown in Fig. 4.

V. Conclusions

We introduced the concept of MIMO TEMs for the representation of multivariate stimuli in the time domain. MIMO TEMs can serve as models for video neural population encoding. We showed how information is represented in the time domain and provided necessary, and practically sufficient, conditions on the spike density as well as on the structure of the filtering kernel that guarantee the representation to be reversible. Interestingly, our MIMO architecture can also serve as a measurement circuit for sub-Nyquist sampling of sparse signals in the frequency domain. The work presented here raises a number of theoretical questions (e.g., frames for vector-valued bandlimited functions, bounds on the number of spikes needed for compressed sensing) and practical issues (e.g., modeling of more complex dendritic tree mechanisms or accounting for other forms of sparsity). These and other issues including neural circuits with random elements [11] will be addressed elsewhere.

Appendix A
MIMO Encoded Vector-Valued Bandlimited Stimuli

By letting $S \to \infty$ we extend the necessary conditions derived in section III to the case of the infinite dimensional space of bandlimited functions. In what follows we denote by Π a subset of the set $\{1, 2, \dots, N\}$, and by Π^c its complement. Moreover for a matrix $\mathbf{A}$ let $\mathbf{A}_\Pi$ denote its row restriction to Π and $\hat{\mathbf{H}}$ denotes the Fourier transform of $\mathbf{H}$. Since the number of spikes becomes infinite, we denote by $D^j = \lim_{S\to\infty} n_j/T$ the average firing rate (spike density) of neuron j.

Theorem 2. *A necessary condition for perfect recovery is that for every subset* $\Pi \subseteq \{1, 2, \dots, N\}$,

$$\sum_{j\in\Pi} \min\left(D^j, R(\{j\})\right) \geq M\frac{\Omega}{\pi} - R(\Pi^c), \tag{18}$$

where $R(\Pi) \triangleq \frac{1}{2\pi}\int_{-\Omega}^{\Omega} r(\hat{\mathbf{H}}_{\mathbf{\Pi}}(\omega))d\omega$.

Proof: We first note that

$$\lim_{S\to\infty} \frac{1}{T} r(\tilde{\mathbf{H}}^{\mathbf{j}}) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} r\left(\hat{\mathbf{H}}^{\mathbf{j}}(\omega)\right) d\omega.$$

Conditions (12) can be relaxed to the spiking conditions:

$$\sum_{j\in\Pi} \min(n_j, r(\tilde{\mathbf{H}}^{\mathbf{j}})) \geq M(2S+1) - \sum_{s=-S}^{S} r(\mathbf{H}_{\mathbf{s}}^{\mathbf{\Pi^c}}). \tag{19}$$

By dividing both sides by T and letting $S \to \infty$ (18) follows. These conditions improve the ones derived in [12] as the latter did not include the rank term on the left side. ■

Remark 3. *A case that is of particular interest to neuroscience is when* $N \gg M$*. Let* Π *be a subset with* $|\Pi| < N - M$*. Assuming that the matrix* $\mathbf{H}_{\mathbf{\Pi^c}}$ *is of full rank for all* ω*, then the right side of* (18) *becomes 0. This suggests that if the population of neurons is very large, then small subsets of the population do not require an actual minimum density.*

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