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Time encoding with an integrate-and-fire neuron with a refractory period

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Abstract

Time encoding is a formal method of mapping amplitude information into a time sequence. We show that under simple conditions, bandlimited stimuli encoded with an integrate-and-fire neuron with an absolute refractory period can be recovered loss-free from the neural spike train at its output. We provide an algorithm for perfect recovery and derive conditions for its convergence.
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1. Introduction

A key question arising in theoretical neuroscience is how to represent an arbitrary stimulus as a sequence of action potentials [1]. The temporal requirements imposed on this representation might dependent on the information presented to the sensory neurons. For example, the temporal precision of auditory processing involves measurements of interaural time delays with sub-millisecond accuracy [3]. Rapid intensity transients appear to be a key stimulus feature for triggering precisely timed spikes [5]. The nervous system uses ensembles of neurons to encode information but direct experimental insights into the operation of biological neural networks is scarce [6].

In [4] the question of stimulus (signal) representation was formulated as one of time encoding, i.e., as one of encoding amplitude information into a time sequence. Formally, a time encoding of a bandlimited function $x = x(t)$, $t \in \mathbb{R}$, is a representation of x as a sequence of strictly increasing times (t_k), $k \in \mathbb{Z}$, where $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and integers, respectively.

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There are two natural requirements that a time encoding mechanism should satisfy. The first is that the encoding should be implemented as a real-time *asynchronous circuit*. Secondly, the encoding mechanism should be *invertible*, that is, the amplitude information can be recovered from the time sequence with arbitrary accuracy. A time encoding machine (TEM) is the realization of such an encoding mechanism.

The first example of a TEM satisfying the above requirements was given [4]. It consists of a feedback loop that contains an adder, a linear filter and a noninverting Schmitt trigger. The invertibility property of the TEM is due to a representation of the bandlimited function $x(t)$, $t \in \mathbb{R}$, as a discrete set of integral values $\int_{t_k}^{t_{k+1}} x(u)\,\mathrm{d}u$. This representation is invertible (or loss-free) provided that the difference between any two consecutive values of the time sequence is bounded by the inverse of the Nyquist rate. Hence, under simple conditions, bandlimited signals encoded with the TEM can be recovered loss-free from the time sequence at its output. A time decoding machine (TDM) is the realization of an algorithm for signal recovery with arbitrary accuracy.

In this paper we investigate a TEM consisting of an integrate-and-fire neuron that incorporates an absolute refractory period. Although the refractory period seems intuitively to lead to information loss, we shall demonstrate that under simple conditions, bandlimited stimuli encoded with an integrate-and-fire neuron with a refractory period can be recovered loss-free from the neural spike train at its output.

This paper is organized as follows: Time encoding and representation of bandlimited stimuli using an integrate-and-fire neuron is analyzed in Section 2. In Section 3 we present the perfect recovery algorithm in its operator and matrix formulations. Section 4 concludes the paper.

2. Time encoding and representation

In this section we introduce a TEM consisting of an integrate-and-fire neuron with an absolute refractory period and describe its key operational properties (see Fig. 1).

The integrator constant κ, the threshold δ, the bias b in Fig. 1 are strictly positive real numbers; $x = x(t)$, $t \in \mathbb{R}$, is a Lebesgues measurable function of finite energy on $\mathbb{R}$ that models the input stimulus to the TEM. Furthermore, x is bounded, $|x(t)| \leqslant c < b$, and bandlimited to $[-\Omega, \Omega]$. The output of the integrator in a small neighborhood

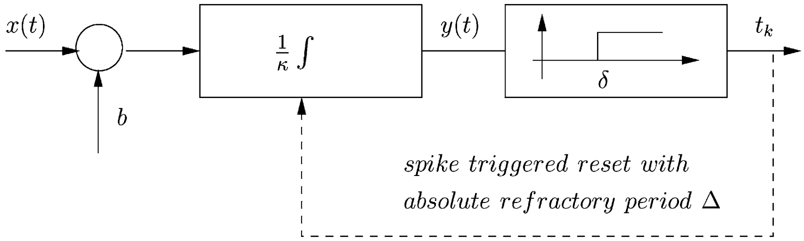

Fig. 1. Time encoding with the integrate-and-fire neuron.

of $t_0 + \Delta$, $t > t_0 + \Delta$ is given by

$$y(t) = y(t_0) + \frac{1}{\kappa}\int_{t_0+\Delta}^{t} [x(u) + b]\,\mathrm{d}u. \tag{1}$$

Note that, due to the bias b, $y = y(t)$ is a continuously increasing function. The output of the TEM is a time sequence (t_k), $k \in \mathbb{Z}$, that models the spike train. The t_k's are also called trigger times.

Lemma 1 (t-Transform). *For all input stimuli $x = x(t)$, $t \in \mathbb{R}$, with $|x(t)| \leqslant c < b$, the output of the TEM is a strictly increasing set of trigger times (t_k), $k \in \mathbb{Z}$, that satisfy the recursive equation*

$$\int_{t_k+\Delta}^{t_{k+1}} x(u)\,\mathrm{d}u = -b(t_{k+1} - t_k - \Delta) + \kappa\delta, \tag{2}$$

for all k, $k \in \mathbb{Z}$.

Proof. The TEM is described in a small neighborhood of $t_0 + \Delta$, $t > t_0 + \Delta$, by

$$\frac{1}{\kappa}\int_{t_0+\Delta}^{t} [x(u) + b]\,\mathrm{d}u = \delta. \tag{3}$$

Since the left-hand side is a continuously increasing function there exists a time $t = t_1$, $t_0 + \Delta < t_1$, such that the equation above holds. Thus, the (output) sequence of times $(t_k)_{k\in\mathbb{Z}}$, is strictly increasing for all k, $k \in \mathbb{Z}$, and the recursion (2) follows. □

Corollary 1 (upper and lower bounds for trigger times). *For all input stimuli $x = x(t)$, $t \in \mathbb{R}$, with $|x(t)| \leqslant c < b$, the distance between consecutive trigger times t_k and t_{k+1} is given by*

$$\frac{\kappa\delta}{b+c} + \Delta \leqslant t_{k+1} - t_k \leqslant \frac{\kappa\delta}{b-c} + \Delta, \tag{4}$$

for all k, $k \in \mathbb{Z}$.

Proof. Since $|x(t)| \leqslant c$, it is easy to see that

$$-c(t_{k+1} - t_k - \Delta) \leqslant \int_{t_k+\Delta}^{t_{k+1}} x(u)\,\mathrm{d}u \leqslant c(t_{k+1} - t_k - \Delta). \tag{5}$$

By replacing the integral in the inequality above with its value given by Eq. (2) and solving for $t_{k+1} - t_k$ we obtain the desired result. The upper and lower bounds are achieved with a constant input $x(t) = c$ for all t, $t \in \mathbb{R}$. □

3. Time decoding and recovery

In this section we derive an algorithm for perfect recovery of the stimulus x based on the knowledge of the trigger times (t_k), $k \in \mathbb{Z}$. In order to achieve this goal we shall employ the operator $\mathscr{A}$ given by

$$\mathscr{A}x = \sum_{k\in\mathbb{Z}} \int_{t_k+\Delta}^{t_{k+1}} x(u)\,\mathrm{d}u \; g(t-s_k), \tag{6}$$

where $g(t)=\sin(\Omega t)/\pi t$ and $s_k=(t_{k+1}+t_k)/2$. The construct of the operator $\mathscr{A}$ above is highly intuitive. Dirac-delta pulses are generated at times s_k with weight $\int_{t_k+\Delta}^{t_{k+1}} x(u)\,\mathrm{d}u$ and then passed through an ideal low pass filter with unity gain for $\omega\in[-\Omega,\Omega]$ and zero otherwise. The values of $\int_{t_k+\Delta}^{t_{k+1}} x(u)\,\mathrm{d}u$ are available at the TDM through Eq. (2). We have the following:

Proposition 1. *If I is the identity operator*,

$$\|I-\mathscr{A}\| \leqslant r+\varepsilon r+\varepsilon, \tag{7}$$

where $\|\cdot\|$ denotes the norm, $r=(\kappa\delta/(b-c)+\Delta)\Omega/\pi$ and $\varepsilon^2=\Delta/(\kappa\delta/(b+c)+\Delta)$.

Proof. It is easy to see that the operator $\mathscr{A}^*$ defined by

$$\mathscr{A}^*x = \sum_{k\in\mathbb{Z}} x(s_k)\mathscr{P}1_{[t_k+\Delta,t_{k+1})} \tag{8}$$

is the adjoint of $\mathscr{A}$, where $\mathscr{P}$ is the projection operator defined as $\mathscr{P}1_{[t_k,t_{k+1})} = (g*1_{[t_k,t_{k+1})})(t)$ (the latter $*$ denotes the convolution operation) and $1_{[t_k,t_{k+1})}$ is a pulse of unit magnitude on $[t_k,t_{k+1})$ and zero otherwise. We have

$$\begin{aligned}\|x-\mathscr{A}^*x\| &= \left\| x-\sum_{k\in\mathbb{Z}} x(s_k)\mathscr{P}1_{[t_k+\Delta,t_{k+1})}\right\| \leqslant \left\| x-\sum_{k\in\mathbb{Z}} x(s_k)1_{[t_k+\Delta,t_{k+1})}\right\| \\ &= \left\|\sum_{k\in\mathbb{Z}}[x-x(s_k)]1_{[t_k,t_{k+1})}+\sum_{k\in\mathbb{Z}} x(s_k)1_{[t_k,t_k+\Delta)}\right\|. \end{aligned} \tag{9}$$

Since ([2], Proposition 3)

$$\left\|\sum_{k\in\mathbb{Z}}[x-x(s_k)]1_{[t_k,t_{k+1})}\right\| \leqslant r\|x\| \tag{10}$$

and noting that

$$\left\|\sum_{k\in\mathbb{Z}} x(s_k)1_{[t_k,t_k+\Delta)}\right\| \leqslant \varepsilon \left\|\sum_{k\in\mathbb{Z}} x(s_k)1_{[t_k,t_{k+1})}\right\| \tag{11}$$

with $\varepsilon^2=\Delta/(\frac{\kappa\delta}{b+c}+\Delta)$, inequality (9) becomes

$$\|x-\mathscr{A}^*x\| \leqslant r\|x\|+\varepsilon(1+r)\|x\| \tag{12}$$

and, therefore,

$$\|x-\mathscr{A}x\| \leqslant (r+\varepsilon r+\varepsilon)\|x\|. \quad \square \tag{13}$$

Let $x_l=x_l(t)$, $t\in\mathbb{R}$, be a sequence of bandlimited functions defined by the recursion $x_{l+1}=x_l+\mathscr{A}(x-x_l)$, for all l, $l\in\mathbb{Z}$, with the initial condition $x_0=\mathscr{A}x$.

Theorem 1 (Operator formulation). *If $r < (1-\varepsilon)/(1+\varepsilon)$, the operator $\mathscr{A}$ is invertible and the stimulus x can be perfectly recovered as*

$$\lim_{l\to\infty} x_l(t) = x(t) \tag{14}$$

and

$$\|x - x_l\| \leqslant (r + \varepsilon r + \varepsilon)^{l+1}\|x\|. \tag{15}$$

Proof. With r defined as above, the proof closely follows one of Theorem 1 in [4].
Let us define $\mathbf{g} = [g(t-s_k)]^{\mathrm{T}}$, $\mathbf{q} = [\int_{t_k+\Delta}^{t_{k+1}} x(u)\,\mathrm{d}u]$ and $\mathbf{G} = [\int_{t_l+\Delta}^{t_{l+1}} g(u-s_k)\,\mathrm{d}u]$. □

We have the following:

Corollary 2 (Matrix formulation). *If $r < (1-\varepsilon)/(1+\varepsilon)$, the bandlimited stimulus x can be perfectly recovered from $(t_k)_{k\in\mathbb{Z}}$ as*

$$x(t) = \lim_{l\to\infty} x_l(t) = \mathbf{g}\mathbf{G}^{+}\mathbf{q}, \tag{16}$$

where $\mathbf{G}^{+}$ denotes the pseudo-inverse of $\mathbf{G}$. Furthermore, $x_l(t) = \mathbf{g}\mathbf{P}_l\mathbf{q}$, where $\mathbf{P}_l$ is given by $\mathbf{P}_l = \sum_{k=0}^{l}(\mathbf{I}-\mathbf{G})^k$.

Proof. Formally identical to the proof of Theorem 2 in [4]. □

4. Conclusion

We have shown that the information contained in the spike train at the output of an integrate-and-fire neuron with absolute refractory period enables the perfect recovery of bandlimited stimuli. The condition for perfect recovery is rather simple. The requirement that $r < (1-\varepsilon)/(1+\varepsilon)$ in Theorem 1, is equivalent to

$$\frac{\kappa\delta}{b-c} + \Delta < \frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{\pi}{\Omega},$$

i.e., the difference between two consecutive trigger times is bounded by a weighed inverse of the Nyquist rate.

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