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Neurocomputing 65–66 (2005) 401–407

NEUROCOMPUTING

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Multichannel time encoding with integrate-and-fire neurons

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Available online 18 January 2005

Abstract

Time encoding is a mechanism of mapping amplitude information into a time sequence. We show that multichannel time encoding using filter banks and integrate-and-fire neurons provides, under natural conditions, an invertible representation of information, i.e., a sensory stimulus can be recovered from its multidimensional spike train representation loss-free. We describe an algorithm for perfect stimulus recovery and derive conditions that guarantee its convergence.

Keywords: Time encoding; Integrate-and-fire neurons; Filter banks; Canonical models; Perfect recovery

1. Introduction

In [6] the question of sensory stimulus (signal) representation was formulated as one of time mapping, i.e., as one of encoding amplitude information into a time sequence. Formally, a time encoding of a bandlimited function $x(t)$, $t \in \mathbb{R}$, is a representation of $x(t)$ as a sequence of strictly increasing times (t_k), $k \in \mathbb{Z}$. The bandlimited function models the stimulus whereas the time sequence models the spike train.

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0925-2312/$ - see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.neucom.2004.10.034

A time encoding machine (TEM) is a realization of a time encoding mechanism that is both asynchronous and invertible. The first example of a TEM satisfying these requirements was given [6]. It consists of a feedback loop that contains an adder, a linear filter and a non-inverting Schmitt trigger. The TEM investigated in [5] consists of an integrate-and-fire neuron with an absolute refractory period. Although the refractory period seems to lead to information loss, it is shown in [5] that under simple conditions, bandlimited stimuli encoded with an integrate-and-fire neuron can be perfectly recovered from the neural spike train at its output. A time decoding machine (TDM) is the realization of an algorithm for stimulus recovery with arbitrary accuracy.

Receptive fields arising in a number of sensory systems, including the retina [8] and the cochlea [3] have been modeled as bank of filters, with each of the filters feeding a signal into a leaky integrate-and-fire neuron. Such a model represents an instantiation of a multichannel time encoding mechanism that maps the amplitude information of the stimulus into a multidimensional spike train. The model raises a number of questions. The one addressed in this paper is whether such a multichannel time encoding mechanism is invertible and if so, what algorithm achieves perfect stimulus recovery.

We shall demonstrate that multichannel time encoding based on filter banks and leaky integrate-and-fire neurons provides, under certain natural conditions, an equivalent representation of information, i.e., the stimulus $(x(t))$, $t \in \mathbb{R}$, can be recovered loss-free from its multidimensional spike train representation (t_k^m), $k \in \mathbb{Z}$ and $m = 1, 2, \ldots, M$. We describe an algorithm for perfect recovery and give conditions that guarantee its convergence.

This paper is organized as follows. Time encoding with single (leaky) integrate-and-fire neurons is investigated in Section 2. The analysis of time encoding and synthesis of time decoding is presented in Sections 2.1 and 2.2, respectively. A multichannel canonical model for employing filter banks and integrate-and-fire neurons for time encoding and stimulus recovery is briefly presented in Section 3. Section 4 concludes the paper.

2. Time encoding with a leaky integrate-and-fire neuron

The TEM considered in this paper is a leaky integrate-and-fire neuron. It consists of a bias, a linear *RC*-filter and a thresholding device. Its basic operation is very simple. The bounded stimulus $|x(t)| \leqslant c < b$, is biased by a constant amount b before being applied to the linear filter. This bias guarantees that the *RC*-filter's output $y(t)$ is an increasing function of time. When the output of the filter reaches a (time-dependent) threshold value δ, a spike is triggered at time t_k at the output. Immediately thereafter the system is reset to an initial state, assumed here to be $y(t_0)$. Therefore, a spike is triggered when the output of the integrator reaches the triggering mark δ (called a quanta). Using a *signal-dependent* sampling mechanism, the TEM maps the amplitude information of $(x(t))$, $t \in \mathbb{R}$, into timing information (t_k), $k \in \mathbb{Z}$.

2.1. Time encoding and representation

In what follows, we shall assume that $x = x(t),\ t \in \mathbb{R}$, with $|x(t)| \leqslant c < b$, is a finite energy signal on $\mathbb{R}$ bandlimited to $[-\Omega, \Omega]$.

The filter parameters R and C, the neuron threshold δ, the bias b are strictly positive real numbers and $x = x(t)$ is a Lebesgues measurable function that models the input signal to the TEM for all $t,\ t \in \mathbb{R}$. The output of the RC-filter in a small neighborhood of t_0, $t > t_0$ is given by

$$y(t) = y(t_0)\exp\left(-\frac{t-t_0}{RC}\right) + \frac{1}{C}\int_{t_0}^{t}[x(u)+b]\exp\left(-\frac{t-u}{RC}\right)\mathrm{d}u. \tag{1}$$

Note that $y = y(t)$ is a continuous function for all $t, t \geqslant t_0$.

Lemma 1 (*t-Transform*). *For all input signals* $x = x(t),\ t \in \mathbb{R}$, *with* $|x(t)| \leqslant c < b$, *the output of the integrate-and-fire neuron is a strictly increasing set of trigger times* (t_k), $k \in \mathbb{Z}$, *obtained from the recursive equation*

$$\begin{aligned}\int_{t_k}^{t_{k+1}} x(u)\exp\left(-\frac{t_{k+1}-u}{RC}\right)\mathrm{d}u \\ = C(\delta - bR) + C[bR - y(t_0)]\exp\left(-\frac{t_{k+1}-t_k}{RC}\right)\end{aligned} \tag{2}$$

for all $k,\ k \in \mathbb{Z}$, *provided that* $y(t_0) < \delta < (b-c)R$.

Proof. The TEM is described in a small neighborhood of t_0, $t > t_0$, by

$$y(t_0)\exp\left(-\frac{t-t_0}{RC}\right) + \frac{1}{C}\int_{t_0}^{t}[x(u)+b]\exp\left(-\frac{t-u}{RC}\right)\mathrm{d}u = \delta. \tag{3}$$

Since the left-hand side is a continuously increasing function, there exists a time $t = t_1$, $t_0 < t_1$, such that the equation above holds. The largest such t_1 is obtained when $x(t) = -c$, for all $t,\ t \in \mathbb{R}$, and

$$t_1 = t_0 + RC\ln\left[1 - \frac{\delta - y(t_0)}{\delta - (b-c)R}\right]. \tag{4}$$

Note that $t_1 > t_0$ if $y(t_0) < \delta < (b-c)R$. Thus, the (output) sequence of times $(t_k)_{k\in\mathbb{Z}}$, is strictly increasing for all $k,\ k \in \mathbb{Z}$, and recursion (2) follows. □

Corollary 1 (*upper and lower bounds for trigger times*). *For all input signals* $x = x(t),\ t \in \mathbb{R}$, *with* $|x(t)| \leqslant c < b$, *the distance between two consecutive trigger times* t_k *and* t_{k+1} *is given by*

$$0 < RC\ln\left[1 - \frac{\delta - y(t_0)}{\delta - (b+c)R}\right] \leqslant t_{k+1} - t_k \leqslant RC\ln\left[1 - \frac{\delta - y(t_0)}{\delta - (b-c)R}\right] \tag{5}$$

for all $k,\ k \in \mathbb{Z}$ *provided that* $y(t_0) < \delta < (b-c)R$.

Proof. Since $|x(t)| \leqslant c$, it is easy to see that

$$-c\left[1-\exp\left(-\frac{t_{k+1}-t_k}{RC}\right)\right] \leqslant \frac{1}{RC}\int_{t_k}^{t_{k+1}} x(u)\exp\left(-\frac{t_{k+1}-u}{RC}\right)\mathrm{d}u \leqslant c\left[1-\exp\left(-\frac{t_{k+1}-t_k}{RC}\right)\right].$$

By replacing the integral in the inequality above with its value given by Eq. (2) and solving for $t_{k+1}-t_k$ we obtain the desired result. The lower bound is achieved for a constant input $x(t)=c$ for all t, $t \in \mathbb{R}$. □

2.2. Time decoding and recovery

In this section we derive an algorithm for perfect recovery of the stimulus x based on the knowledge of the trigger times (t_k), $k \in \mathbb{Z}$. In order to achieve this goal we shall employ the operator $\mathscr{A}$ given by

$$\mathscr{A}x = \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} x(u)\exp\left(-\frac{t_{k+1}-u}{RC}\right)\mathrm{d}u\, g(t-s_k), \tag{6}$$

where $g(t)=\sin(\Omega t)/\pi t$ and $s_k=(t_{k+1}+t_k)/2$. The construct of the operator $\mathscr{A}$ above is highly intuitive and utilizes the filtered version of $x(t)$ on the interval $[t_k, t_{k+1}]$. Dirac-delta pulses are generated at times s_k with weight

$$\int_{t_k}^{t_{k+1}} x(u)\exp\left(-\frac{t_{k+1}-u}{RC}\right)\mathrm{d}u$$

and then passed through an ideal low pass filter with unity gain for $\omega \in [-\Omega, \Omega]$ and zero otherwise. The values of

$$\int_{t_k}^{t_{k+1}} x(u)\exp\left(-\frac{t_{k+1}-u}{RC}\right)\mathrm{d}u$$

are available at the TDM through Eq. (2).

Lemma 2. *If I is the identity operator*,

$$\|I-\mathscr{A}\| \leqslant r + \varepsilon r + \varepsilon, \tag{7}$$

where $\|\cdot\|$ *denotes the norm and*

$$r = RC\ \ln\left[1-\frac{\delta-y(t_0)}{\delta-(b-c)R}\right]\frac{\Omega}{\pi} \text{ and } \varepsilon = \frac{\delta-y(t_0)}{(b-c)R-y(t_0)}.$$

Proof. Similar to proposition 1 of [5]. □

Let $x_l = x_l(t)$, $t \in \mathbb{R}$, be a sequence of bandlimited functions defined by the recursion:

$$x_{l+1} = x_l + \mathscr{A}(x-x_l), \tag{8}$$

for all l, $l \in \mathbb{Z}$, with the initial condition $x_0 = \mathscr{A}x$.

Proposition 1 (*operator formulation*). *Let* $x = x(t)$, $t \in \mathbb{R}$, *be a bounded signal* $|x(t)| \leqslant c < b$ *bandlimited to* $[-\Omega, \Omega]$. *If* $r < (1-\varepsilon)/(1+\varepsilon)$, *the operator* $\mathscr{A}$ *is invertible and the signal* x *can be perfectly recovered as* $x(t) = \lim_{l\to\infty} x_l(t)$, *and* $\|x - x_l\| \leqslant r^{l+1}\|x\|$.

Proof. Closely follows Theorem 1 in [5]. □

Let us define the vectors

$$\mathbf{g} = [g(t - s_k)]^{\mathrm{T}}, \quad \mathbf{q} = \left[\int_{t_k}^{t_{k+1}} x(u) \exp\left(-\frac{t_{k+1} - u}{RC}\right) \mathrm{d}u\right]$$

and the matrix

$$\mathbf{G} = \left[\int_{t_l}^{t_{l+1}} g(u - s_k) \exp\left(-\frac{t_{l+1} - u}{RC}\right) \mathrm{d}u\right].$$

We have the following:

Corollary 2 (*matrix formulation*). *Under the assumptions of Proposition 1 the bandlimited signal* x *can be perfectly recovered from* $(t_k)_{k\in\mathbb{Z}}$ *as*

$$x(t) = \lim_{l\to\infty} x_l(t) = \mathbf{g}\mathbf{G}^{+}\mathbf{q},$$

where $\mathbf{G}^{+}$ *denotes the pseudo-inverse of* $\mathbf{G}$. *Furthermore*, $x_l(t) = \mathbf{g}\mathbf{P}_l\mathbf{q}$, *where* $\mathbf{P}_l$ *is given by* $\mathbf{P}_l = \sum_{k=0}^{l} (\mathbf{I} - \mathbf{G})^k$.

Proof. Formally identical to the proof of Theorem 2 in [6]. □

3. A canonical model for time encoding

The canonical model for time encoding and decoding consists of an information representation (analysis) subsystem and an information recovery (synthesis) subsystem. The representation subsystem is shown in Fig. 1. It consists of a generic filter bank followed by a cascade of TEMs. The output of each filter is encoded with a TEM modeling the operation of an integrate-and-fire neuron. The recovery subsystem consists of a cascade of TDMs followed by appropriately chosen filters.

The filter banks can be designed using various methodologies. The one considered here is based either on the wavelet transform or the Gabor transform [2]. The conditions for invertibility on the generated filter banks are quite standard. Informally, they only require that overall no signal frequency is lost due to filtering. This does not rule out overcomplete representations. Since under this condition the filter bank representations (e.g., the wavelet and Gabor) are invertible in their own right, the signal can be recovered loss-free from the multidimensional time sequence. Readers unfamiliar with the filter bank representation and recovery formalism are referred to [4].

Filter bank representations of bandlimited signals have been extensively studied in the literature (see, e.g., [1] and the references therein). However, the sampling of the

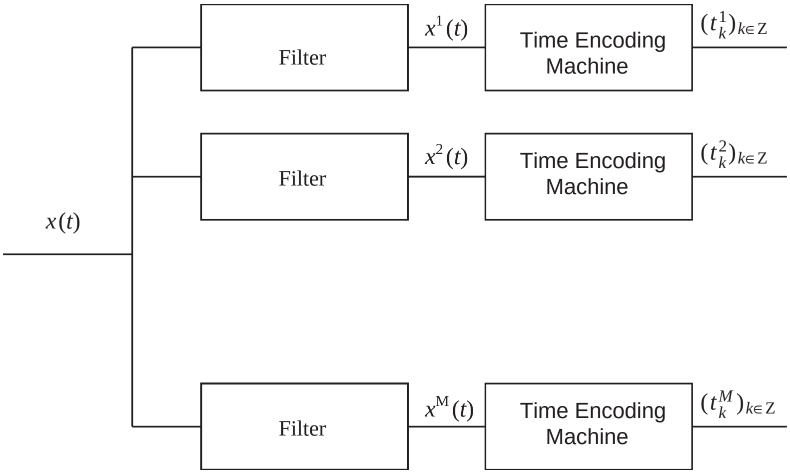

Fig. 1. Canonical model for time encoding.

output of the filter bank is achieved by traditional amplitude sampling and, therefore, the data set for representing stimulus information is substantially different from ours.

4. Conclusions

In this paper we presented a canonical model for time encoding and stimulus recovery for sensory systems. The model consists of a filter bank followed by a cascade of leaky integrate-and-fire neurons. The key advantage of the model is its flexibility in modelling various sensory systems. Under natural conditions, the canonical model is invertible even though the constituent filters have overlapping frequency bands and the integrate-and-fire neurons operate with possibly different threshold values. The invertibility property is remarkable particularly because the individual TEMs are non-linear devices.

The canonical model helps elucidate some of the key open questions of temporal coding for sensory systems. First, stimuli encoded by a single leaky integrate-and-fire neuron can be recovered loss-free from the neural spike train. The recovery of the stimulus requires information about the trigger times. There is no need to repeat an experiment to obtain additional spike train data about the stimulus. For sensory systems using spikes for recovering the stimulus from a single running experiment is a defining biological requirement. Second, the canonical model shows that the same stimulus can be recovered using different filter transfer functions and integrate-and-fire neuron parameters. The latter result seems to be particularly noteworthy because the choice of bounded thresholding functions leads to different representations of the stimulus without information loss. An algorithm that performs perfect recovery and is insensitive with respect to the value of the threshold appears in [7].

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